

SU-GP-93/4-5
gr-qc/9305003

Holomorphic quantum mechanics with a quadratic Hamiltonian constraint

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(May 1993)

Abstract

A finite dimensional system with a quadratic Hamiltonian constraint is Dirac quantized in holomorphic, antiholomorphic and mixed representations. A unique inner product is found by imposing Hermitian conjugacy relations on an operator algebra. The different representations yield drastically different Hilbert spaces. In particular, all the spaces obtained in the antiholomorphic representation violate classical expectations for the spectra of certain operators, whereas no such violation occurs in the holomorphic representation. A subset of these Hilbert spaces is also recovered in a configuration space representation. A propagation amplitude obtained from an (anti)holomorphic path integral is shown to give the matrix elements of the identity operators in the relevant Hilbert spaces with respect to an overcomplete basis of representation-dependent generalized coherent states. Relation to quantization of spatially homogeneous cosmologies is discussed in view of the no-boundary proposal of Hartle and Hawking and the new variables of Ashtekar.

Pacs: 04.60.+n, 03.65.Fd, 98.80.Bp

arXiv:gr-qc/9305003 v1 3 May 1993

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I. INTRODUCTION

The three standard formulations of quantum theory are Schrödinger's wave function formulation, Heisenberg's operator formulation, and Feynman's path integral formulation. Although these methods can be considered equivalent in ordinary nonrelativistic quantum mechanics in their common domain of validity, there exist systems of interest in which the applicability and interrelation of these methods is less clear. An example that has recently received considerable attention is quantum field theory on spacetimes containing closed timelike loops [1]. Another example, on which we shall concentrate in this paper, are Hamiltonian systems whose dynamics is entirely generated by constraints. An important class of such systems are reparametrization invariant theories, including the general theory of relativity.

An analogue of the Schrödinger method of quantization for constrained Hamiltonian systems was developed by Dirac [2], the idea being to promote the (first class) classical constraints into quantum operators that are postulated to annihilate the wave function. For systems whose dynamics is entirely generated by constraints this method can be regarded as coinciding with the Heisenberg method, as such systems possess no preferred, external time variable. The Dirac method in the form given in Ref. [2] is, however, generally considered incomplete in that it does not specify an inner product, or any other means of obtaining the probabilistic predictions one expects of a quantum theory [3,4]. Methods for providing such an inner product have been proposed in Refs. [3,5,6].

On the other hand, path integral quantization methods have been extensively applied to constrained Hamiltonian systems, and there is ample evidence [7–9] that appropriately defined path integrals are related to the constraint equations of Dirac quantization. Apart from simple exactly solvable examples, such as the relativistic point particle [10,11], it appears however unclear what the relation of such path integrals should be to any Hilbert space structure that one may have introduced in the Dirac quantization. A recent discussion of these issues is given in Ref. [12].

In much of the work on path integrals in constrained systems, the attention has been on (extended) configuration space or phase space path integrals. In this paper we shall consider a constrained system which admits a natural class of coherent state, or “holomorphic,” representations in the Dirac quantization, as well as a natural class of coherent state path integral constructions. Our first aim is to analyze the quantum theories that are obtained through the Dirac method in the different coherent state representations, adopting Ashtekar's suggestion that the inner product be determined from the Hermitian conjugacy relations on a suitable operator algebra [3]. Our second aim is to establish that certain coherent state path integrals do produce quantum amplitudes that can be interpreted both as operators and as generalized coherent state vectors in the Hilbert spaces emerging from the Dirac quantization.

Our model consists of two harmonic oscillators with identical frequencies, with the dynamics given by a single Hamiltonian constraint which sets the energy difference of the oscillators equal to a prescribed real number. The constraint is analogous to the Hamiltonian constraint in general relativity [13] in that both consist of a sum of a quadratic kinetic term and a potential term, and further in that the kinetic and potential terms have indefinite sign. Models of this kind arise from spatially homogeneous cosmologies that admit a Hamil-

tonian formulation [14–17], and there exists a class of spatially homogeneous cosmologies that are, modulo certain global caveats, exactly described by our model [18,19].

The classical properties of this and related models were extensively analyzed in Refs. [20,21] from the viewpoint of topological obstructions to quantization, and difficulties of the Dirac quantization in the position representation were discussed in [21]. The Dirac quantization of this model in Ashtekar’s program has been previously considered in Refs. [3,22] in a representation based on generalized angular momentum eigenstates.

In the Dirac quantization, we shall consider a class of representations which generalize the Bargmann representation for a single unconstrained harmonic oscillator [23,24]. The quantum states are represented by analytic or antianalytic functions of the “complex coordinates” of each oscillator, providing a total of four different possibilities (analytic in both arguments, antianalytic in both arguments, or analytic in one argument and antianalytic in the other). The inner product will be uniquely determined, under certain assumptions, by imposing Hermitian conjugacy relations on a suitable operator algebra. Unlike in the case of a single unconstrained harmonic oscillator, all the four representations will lead into sets of Hilbert spaces. While some of these sets overlap in part, the intersection of all four sets is empty, and the individual quantum theories have drastically different properties. In particular, in all the quantum theories obtained in the representation where the wave functions are antianalytic in both oscillators, the spectra of certain operators violate inequalities satisfied by their classical counterparts. Conversely, no such violation occurs in the representation where the wave functions are analytic in both oscillators.

We also discuss the Dirac quantization in a position representation in which the quantum states are represented by functions of the real-valued configuration space coordinates. In this representation one recovers Hilbert spaces that are isomorphic to those above mentioned ones in which the operators do not violate the inequalities satisfied by their classical counterparts.

Coherent state path integrals are constructed in analogy with those for the unconstrained harmonic oscillator [24–26]. With a careful definition, the path integral is shown to yield the matrix elements of the identity operator in the Hilbert spaces obtained in Dirac’s quantization, with respect to an overcomplete basis of representation-dependent generalized coherent states. In the representations where the wave functions are analytic or antianalytic in both arguments, this basis consists of simultaneous annihilation operator and minimum uncertainty coherent states associated with the Lie algebra of $SO(2,1)$ [26]. In the mixed representations, on the other hand, the basis consists of displacement operator coherent states associated with the same Lie algebra [26,27].

Using the relation of our model to spatially homogeneous cosmologies, the above results are reinterpreted as quantizations of certain cosmological models [18,19]. In the position representation, we show that the “ground state” wave function in the relevant Hilbert space has the semiclassical behavior conventionally associated with the no-boundary wave function of Hartle and Hawking [28–30]. The same holds also in the corresponding coherent state representation. A path integral definition of the no-boundary wave function beyond the semiclassical estimate remains however problematic in all the representations.

The paper is organized as follows. In section II we introduce the model and review its classical dynamics in a set of complex phase space variables. In section III we carry out the Dirac and path integral quantizations in the representation where the wave functions are analytic in both arguments, and outline the corresponding results for the representation

where the wave functions are antianalytic in both arguments. Section IV contains the corresponding analysis in the mixed representations. The Dirac quantization in the position representation is carried out in section V, and the cosmological interpretation is discussed in section VI.

The results are summarized and discussed in section VII. We in particular discuss the relation of our approach to the quantization of spatially homogeneous cosmologies in the connection representation of Ashtekar's variables [3], and to the Born-Oppenheimer expansion in quantum cosmology. A related model in which the energy difference constraint is replaced by an energy sum constraint is briefly analyzed in the appendix.

II. THE MODEL

We consider a model whose action is

$$S = \int dt (p_1 \dot{x}_1 + p_2 \dot{x}_2 - N\mathcal{H}) \quad , \quad (2.1)$$

where the constraint \mathcal{H} is given by

$$\mathcal{H} = \frac{1}{2}(p_1^2 - p_2^2 + x_1^2 - x_2^2) - 2\delta \quad , \quad (2.2)$$

and δ is an arbitrary real number. The unconstrained phase space Γ is \mathbf{R}^4 , with global canonical coordinates (x_I, p_I) , $I = 1, 2$, with Poisson brackets $\{x_I, p_I\} = \delta_{IJ}$. N is a Lagrange multiplier enforcing the constraint.

Physically, the model describes two harmonic oscillators with identical frequencies and an energy difference equal to 2δ . The classical equations of motion are easily solved. As a preparation to the quantization, we shall in this section describe the classical dynamics in terms of a complex set of functions on the phase space.

To begin, define on Γ the complex-valued functions

$$\begin{aligned} z_I &= \frac{1}{\sqrt{2}}(x_I - ip_I) \\ \bar{z}_I &= \frac{1}{\sqrt{2}}(x_I + ip_I) \quad . \end{aligned} \quad (2.3)$$

The set $\mathcal{S} = \{z_I, \bar{z}_I, 1\}$ of functions on Γ is closed under the Poisson bracket, $\{z_I, \bar{z}_J\} = i\delta_{IJ}$, and every sufficiently regular function on Γ can be expressed in terms of (possibly infinite) sums and products of the elements of \mathcal{S} . The constraint takes the form

$$\mathcal{H} = z_1 \bar{z}_1 - z_2 \bar{z}_2 - 2\delta \quad . \quad (2.4)$$

The classical Poisson bracket algebra generated by the elements in \mathcal{S} is therefore sufficiently large for describing the classical dynamics of the system. This algebra will be used as the starting point of quantization in the next section.

Next, we need a set of constants of motion. Consider on Γ the three functions

$$J_+ = z_1 z_2 \quad (2.5a)$$

$$J_- = \bar{z}_1 \bar{z}_2 \quad (2.5b)$$

$$J_0 = \frac{1}{2}(z_1 \bar{z}_1 + z_2 \bar{z}_2) \quad (2.5c)$$

whose Poisson brackets form a closed algebra by

$$\begin{aligned}\{J_+, J_-\} &= 2iJ_0 \\ \{J_0, J_\pm\} &= \mp iJ_\pm \quad .\end{aligned}\tag{2.6}$$

As the J 's have (strongly) vanishing Poisson brackets with \mathcal{H} , their values on the constraint surface $\mathcal{H} = 0$ are constants of motion. It is straightforward to verify that the J 's are a complete set of constants of motion, in the sense that every classical solution is uniquely specified by the values of the J 's.

On the constraint surface $\mathcal{H} = 0$, the J 's are not algebraically independent but satisfy the identity

$$-J_0^2 + J_+J_- = -\delta^2 \quad .\tag{2.7}$$

An algebraically independent set of constants of motion that is large enough to uniquely specify a classical solution is given by the real and imaginary parts of J_\pm . These two real numbers can take arbitrary values, and they determine J_0 uniquely as the non-negative root of Eq. (2.7). The space of constants of motion, denoted by $\bar{\Gamma}$, is therefore topologically \mathbf{R}^2 .

For $\delta \neq 0$ the constraint surface is a manifold, and $\bar{\Gamma}$ inherits from Γ a natural differentiable structure in which a global coordinate chart is provided by the real and imaginary parts of J_\pm . The symplectic form $\Omega = \sum_I dp_I \wedge dx_I$ on Γ has a smooth non-degenerate pull-back to $\bar{\Gamma}$, given by

$$\Omega_{\bar{\Gamma}} = \frac{i dJ_+ \wedge dJ_-}{2J_0} \quad ,\tag{2.8}$$

where J_0 is the positive root of (2.7). For $\delta = 0$ the constraint surface is not a manifold near the point $x_I = p_I = 0$, but there nevertheless exists a differentiable structure on $\bar{\Gamma}$ such that the pull-back of Ω is smooth and non-degenerate. A global coordinate chart in this differentiable structure is provided by the real and imaginary parts of the functions $I_\pm = J_\pm(J_+J_-)^{-1/4}$, and one has

$$\Omega_{\bar{\Gamma}} = i dI_+ \wedge dI_- \quad .\tag{2.9}$$

$\bar{\Gamma}$ has therefore the structure of a genuine reduced phase space for all values of δ .

III. QUANTUM THEORY IN THE DOUBLY HOLOMORPHIC REPRESENTATION

We now embark on quantization of the model. In subsection III A we apply the Dirac method, following the algebraic quantization program of Ref. [3], and choose a representation analogous to the Bargmann representation of an unconstrained harmonic oscillator [23,24]. In subsection III B we shall relate the resulting Hilbert spaces to path integral quantization.

A. Doubly holomorphic Dirac quantization

We begin by introducing a set of elementary quantum operators, $\hat{\mathcal{S}} = \{\hat{z}_I, \hat{\bar{z}}_I, \hat{\mathbb{1}}\}$, with the commutator algebra

$$\begin{aligned} [\hat{z}_I, \hat{\bar{z}}_J] &= -\delta_{IJ} \hat{\mathbb{1}} \\ [\hat{z}_I, \hat{\mathbb{1}}] &= [\hat{\bar{z}}_I, \hat{\mathbb{1}}] = 0 \quad . \end{aligned} \quad (3.1)$$

Here, and from now on, $\hat{\mathbb{1}}$ stands for the identity operator. The set $\hat{\mathcal{S}}$ and its commutator algebra are the quantum counterparts of the set \mathcal{S} and its Poisson bracket algebra. The full quantum operator algebra \mathcal{A} is the algebra generated by $\hat{\mathcal{S}}$.

We wish to represent \mathcal{A} on a vector space (“space of wave functions”). We choose the space of complex analytic functions in two variables,

$$V_{\text{HH}} = \{\psi(z_I) \mid \psi \text{ analytic in } z_I\} \quad , \quad (3.2)$$

with the representation

$$\begin{aligned} \hat{z}_I \psi &= z_I \psi \\ \hat{\bar{z}}_I \psi &= \frac{\partial \psi}{\partial z_I} \quad . \end{aligned} \quad (3.3)$$

In analogy with the Bargmann representation of the unconstrained harmonic oscillator [23,24], we refer to this as the doubly holomorphic representation. (Note that our conventions for the barred and unbarred quantities agree with those of Ref. [23] and are opposite of those of Ref. [24].) It is not necessary at this stage to be specific about the allowed singularities in the functions in V_{HH} . We shall eventually consider subspaces of V_{HH} where the functions will have precisely defined analyticity properties.

The operator version of the classical Hamiltonian constraint \mathcal{H} (2.4) is taken to be

$$\hat{\mathcal{H}} = \hat{z}_1 \hat{\bar{z}}_1 - \hat{z}_2 \hat{\bar{z}}_2 - 2\delta \hat{\mathbb{1}} \quad . \quad (3.4)$$

This can be thought of as ordering the barred operators to the right of the unbarred ones. However, as the two oscillators enter \mathcal{H} with the opposite signs, any ordering of the form $z_I \bar{z}_I \mapsto (1-t)\hat{z}_I \hat{\bar{z}}_I + t\hat{\bar{z}}_I \hat{z}_I$, with the same t for both oscillators, would give the same result.

The subspace of V_{HH} where the quantum constraint equation $\hat{\mathcal{H}}\psi = 0$ is satisfied is easily found. It is

$$V_{\text{HH},0} = \left\{ \psi \in V_{\text{HH}} \mid \psi(z_I) = (z_1/z_2)^\delta \phi(z_1 z_2) \right\} \quad , \quad (3.5)$$

where ϕ is a complex analytic function of its single argument. Again, it is not necessary at this stage to be specific about the singularities in ϕ . Note that solving the constraint has introduced no assumptions about the separability or normalizability of the wave functions.

We now wish to build an algebra of physical operators, that is, a subalgebra of \mathcal{A} that would leave $V_{\text{HH},0}$ invariant. To do this, notice first that the Poisson bracket algebra of the classical J 's (2.5) can be promoted into a commutator algebra by setting

$$\hat{J}_+ = \hat{z}_1 \hat{z}_2 \quad (3.6a)$$

$$\hat{J}_- = \hat{\bar{z}}_1 \hat{\bar{z}}_2 \quad (3.6b)$$

$$\hat{J}_0 = \frac{1}{2} \left(\hat{z}_1 \hat{\bar{z}}_1 + \hat{z}_2 \hat{\bar{z}}_2 + \hat{\mathbb{1}} \right) , \quad (3.6c)$$

so that the commutators are

$$\begin{aligned} [\hat{J}_+, \hat{J}_-] &= -2\hat{J}_0 \\ [\hat{J}_0, \hat{J}_\pm] &= \pm \hat{J}_\pm . \end{aligned} \quad (3.7)$$

(Once \hat{J}_+ and \hat{J}_- are chosen as in (3.6), the term proportional to $\hat{\mathbb{1}}$ in \hat{J}_0 is fixed by requiring that the commutator algebra closes.) Eqs. (3.7) are recognized as the commutators of the Lie algebra of $SO(2,1)$. As all the \hat{J} 's commute with $\hat{\mathcal{H}}$, they leave $V_{\text{HH},0}$ invariant. We therefore choose our physical operator algebra \mathcal{A}_{phy} to be the algebra generated by the set $\{\hat{J}_\pm, \hat{J}_0, \hat{\mathbb{1}}\}$. This algebra is made into a star-algebra $\mathcal{A}_{\text{phy}}^{(*)}$ by introducing the star-relation (involution)

$$\hat{J}_\pm^* = \hat{J}_\mp , \quad \hat{J}_0^* = \hat{J}_0 , \quad \hat{\mathbb{1}}^* = \hat{\mathbb{1}} \quad (3.8)$$

which is inherited from the complex conjugation relations of the classical J 's.

As the classical J 's provide an overcomplete set of constants of motion, we expect $\mathcal{A}_{\text{phy}}^{(*)}$ to be a sufficiently large algebra for the construction of a quantum theory. However, in analogy with the algebraic identity (2.7) satisfied by the J 's on the constraint surface, the quantum \hat{J} 's on $V_{\text{HH},0}$ are not independent but satisfy the algebraic relation

$$-\hat{J}_0^2 + \frac{1}{2} (\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+) = \left(\frac{1}{4} - \delta^2 \right) \hat{\mathbb{1}} . \quad (3.9)$$

The left hand side of Eq. (3.9) is recognized as the Casimir invariant of $SO(2,1)$ [31]. This means that our representation of the Lie algebra of $SO(2,1)$ on $V_{\text{HH},0}$ contains only those irreducible representations where the value of the Casimir invariant is $\frac{1}{4} - \delta^2$. The shift from the classical value $-\delta^2$ on the right hand side of Eq. (2.7) can be traced to our factor ordering of the constraint and, at a deeper level, to the relation between reduced phase space quantization and the Dirac quantization [22,32].

Writing the vectors in $V_{\text{HH},0}$ as in Eq. (3.5), in terms of a function of a single complex variable, the action of the \hat{J} 's takes the form

$$(\hat{J}_+ \phi)(w) = w \phi(w) \quad (3.10a)$$

$$(\hat{J}_- \phi)(w) = w \phi''(w) + \phi'(w) - \frac{\delta^2}{w} \phi(w) \quad (3.10b)$$

$$(\hat{J}_0 \phi)(w) = w \phi'(w) + \frac{1}{2} \phi(w) , \quad (3.10c)$$

where the prime denotes derivative with respect to the argument w . From this it is clear that the representation of $\mathcal{A}_{\text{phy}}^{(*)}$ on $V_{\text{HH},0}$ is highly reducible. Consider therefore the subspace where ϕ is a linear combination of arbitrary powers,

$$\tilde{V}_{\text{HH},0} = \text{Span} \left\{ \psi_m \in V_{\text{HH},0} \mid \psi_m(z_I) = (z_1/z_2)^\delta (z_1 z_2)^m, m \in \mathbf{C} \right\} . \quad (3.11)$$

The action of the \hat{J} 's on the basis vectors ψ_m is

$$\hat{J}_+ \psi_m = \psi_{m+1} \quad (3.12a)$$

$$\hat{J}_- \psi_m = (m^2 - \delta^2) \psi_{m-1} \quad (3.12b)$$

$$\hat{J}_0 \psi_m = \left(m + \frac{1}{2}\right) \psi_m \quad , \quad (3.12c)$$

and $\tilde{V}_{\text{HH},0}$ therefore carries a representation of $\mathcal{A}_{\text{phy}}^{(*)}$. This representation is still reducible, but all the subspaces of $\tilde{V}_{\text{HH},0}$ carrying irreducible representations are straightforward to find. All these irreducible representations are cyclic, with some $\psi_m \in \tilde{V}_{\text{HH},0}$ as the cyclic vector. They have countably infinite dimension, by virtue of Eq. (3.12a), and they are all faithful.

(Note that we have not specified a topology on $\tilde{V}_{\text{HH},0}$ or on its subspaces. By irreducibility we therefore understand algebraic irreducibility, which means nonexistence of nontrivial invariant subspaces. It has been argued that when representing an operator algebra on a Hilbert space, the physically appropriate criterion is topological irreducibility, which means nonexistence of nontrivial closed invariant subspaces [33].)

To complete the quantization, we wish to introduce an inner product. Given a subspace of $\tilde{V}_{\text{HH},0}$ on which the representation of $\mathcal{A}_{\text{phy}}^{(*)}$ is irreducible, we seek on this subspace an inner product such that the \hat{J} 's satisfy the Hermitian conjugacy relations

$$\hat{J}_\pm^\dagger = \hat{J}_\mp \quad , \quad \hat{J}_0^\dagger = \hat{J}_0 \quad (3.13)$$

which correspond to the star-relation (3.8) of $\mathcal{A}_{\text{phy}}^{(*)}$. If such an inner product exists, a Hilbert space is constructed by Cauchy completion.

It turns out that an inner product satisfying these requirements exists only for a subset of the irreducible representations; however, when such an inner product exists, it is unique (up to an overall constant). The proof of these assertions is straightforward. The Hermiticity of \hat{J}_0 and Eq. (3.12c) first tell that the index m is real and the inner product is diagonal in the basis $\{\psi_m\}$. The Hermitian conjugacy of \hat{J}_\pm and Eqs. (3.12a)-(3.12b) then relate the norms of ψ_m for different m , and the requirement that the inner product be positive definite sets the final restriction on the values of m . There are just three different cases, which we now list in turn.

For any $\delta \in \mathbf{R}$, one admissible vector space is

$$V_{\text{HH}}^+ = \text{Span} \left\{ \psi_m \in \tilde{V}_{\text{HH},0} \mid m = |\delta| + k, k = 0, 1, 2, \dots \right\} \quad . \quad (3.14)$$

The inner product is

$$(\psi_m, \psi_{m'}) = \Gamma(m+1+\delta)\Gamma(m+1-\delta)\delta_{m,m'} \quad (3.15)$$

where $\delta_{m,m'}$ stands for the Kronecker delta. To examine the physical content of this space, consider the normal ordered energy operators for each oscillator,

$$\hat{E}_I = \hat{z}_I \hat{\bar{z}}_I + \frac{1}{2} \hat{\mathbb{1}} \quad . \quad (3.16)$$

These operators clearly are in $\mathcal{A}_{\text{phy}}^{(*)}$, and the basis vectors $\psi_{|\delta|+k}$ are joint eigenvectors of \hat{E}_I . The eigenenergies of the lower energy oscillator (which is the first oscillator for $\delta < 0$ and the second oscillator for $\delta > 0$) are $k + \frac{1}{2}$, that is, exactly the positive half-integers that one would anticipate on the basis of a single unconstrained oscillator. The respective eigenenergies of the higher energy oscillator are $k + \frac{1}{2} + 2|\delta|$, which are half-integers only if 2δ is an integer. In the degenerate case $\delta = 0$ the eigenenergies of the two oscillators coincide. The spectrum of \hat{J}_0 is bounded below by $|\delta| + \frac{1}{2}$, and this space therefore respects the classical inequality $J_0 \geq |\delta|$.

If $0 < |\delta| < \frac{1}{2}$, a second admissible vector space is

$$V_{\text{HH}}^- = \text{Span} \left\{ \psi_m \in \tilde{V}_{\text{HH},0} \mid m = -|\delta| + k, k = 0, 1, 2, \dots \right\} . \quad (3.17)$$

The inner product is again given by Eq. (3.15). Again, the basis vectors $\psi_{-|\delta|+k}$ are simultaneous eigenvectors of \hat{E}_I . Now the eigenenergies of the higher energy oscillator are the positive half-integers $k + \frac{1}{2}$, and those of the lower energy oscillator are respectively $k + \frac{1}{2} - 2|\delta|$, which are never half-integers. The spectrum of \hat{J}_0 is bounded below by $\frac{1}{2} - |\delta|$. The classical inequality $J_0 \geq |\delta|$ is therefore violated if $\frac{1}{4} < |\delta| < \frac{1}{2}$, but only moderately. Note that the inequality $\frac{1}{4} < |\delta| < \frac{1}{2}$ is precisely the condition under which the lowest eigenenergy of the lower energy oscillator is negative.

Finally, if $|\delta| < \frac{1}{2}$, there exists a one-parameter family of admissible vector spaces, labeled by a real number ϵ satisfying $|\delta| < \epsilon < 1 - |\delta|$. These spaces are

$$V_{\text{HH}}^\epsilon = \text{Span} \left\{ \psi_m \in \tilde{V}_{\text{HH},0} \mid m = \epsilon + k, k \in \mathbf{Z} \right\} , \quad (3.18)$$

and the inner product is again given by Eq. (3.15). The spectra of \hat{J}_0 and \hat{E}_I are unbounded both above and below.

These three cases give the complete list of the inner product spaces. The respective abstract Hilbert spaces $\mathcal{H}_{\text{HH}}^+$, $\mathcal{H}_{\text{HH}}^-$ and $\mathcal{H}_{\text{HH}}^\epsilon$ are obtained through Cauchy completion. The factorial growth of the inner product (3.15) at $m \rightarrow \infty$ guarantees that the vectors in $\mathcal{H}_{\text{HH}}^\pm$ can be represented by complex analytic functions of the form given in Eq. (3.5): the function $\phi(w)$ is such that $w^{\mp|\delta|}\phi(w)$ (where the upper and lower signs correspond to $\mathcal{H}_{\text{HH}}^\pm$, respectively) is analytic everywhere in the finite complex w plane. The spaces $\mathcal{H}_{\text{HH}}^\epsilon$, on the other hand, all contain vectors that cannot be represented by complex analytic functions of the form given in Eq. (3.5), the reason being the inverse factorial decay of the inner product (3.15) for $m = \epsilon + k$ at $k \rightarrow -\infty$.

Therefore, if we require that all vectors in the Hilbert space be representable by complex analytic functions of the form given in Eq. (3.5), only the spaces $\mathcal{H}_{\text{HH}}^\pm$ remain. The inner product of any $\psi_{(1)}, \psi_{(2)} \in \mathcal{H}_{\text{HH}}^\pm$ can be written as the holomorphic integral

$$(\psi_{(1)}, \psi_{(2)}) = \int \frac{d\bar{w} \wedge dw}{\pi i} K_{2\delta} \left(2\sqrt{w\bar{w}} \right) \overline{\phi_{(1)}(w)} \phi_{(2)}(w) , \quad (3.19)$$

where $d\bar{w} \wedge dw = 2i \text{Re}(w) \wedge \text{Im}(w)$, the integration is over the whole complex w plane, and $K_{2\delta}$ is a modified Bessel function [34].

We have thus recovered a set of Hilbert spaces carrying representations of the $SO(2, 1)$ Lie algebra. This set contains a continuum of spaces in which the spectrum of \hat{J}_0 is unbounded both above and below, as well as one or two spaces, depending on the value of δ ,

in which the spectrum of \hat{J}_0 is bounded below but not above. From the viewpoint of the $SO(2, 1)$ Lie algebra (3.7) alone, this result might seem surprising. The Lie algebra admits an automorphism which interchanges \hat{J}_+ with \hat{J}_- and reverses the sign of \hat{J}_0 : for any representation of the algebra, there must therefore exist another representation with an inverted spectrum of \hat{J}_0 . The reason why our set of representations does not exhibit such a symmetry is in our choice of the doubly holomorphic representation. Whereas the algebra (3.7) tells that there has to be a total factor of $(m^2 - \delta^2)$ in the actions of \hat{J}_\pm in Eqs. (3.12), it is the choice of the doubly holomorphic representation that distributes this factor unsymmetrically between the raising and lowering operators.

Suppose that, instead of the doubly holomorphic representation, one adopts a doubly antiholomorphic representation in which the algebra \mathcal{A} is represented on the vector space

$$V_{AA} = \left\{ \tilde{\psi}(\bar{z}_I) \mid \tilde{\psi} \text{ antianalytic in } \bar{z}_I \right\} , \quad (3.20)$$

by

$$\begin{aligned} \hat{z}_I \tilde{\psi} &= -\frac{\partial \tilde{\psi}}{\partial \bar{z}_I} \\ \hat{\bar{z}}_I \tilde{\psi} &= \bar{z}_I \tilde{\psi} . \end{aligned} \quad (3.21)$$

The analysis can be carried through in perfect analogy with the above. The counterpart of Eqs. (3.12) is now

$$\hat{J}_+ \tilde{\psi}_m = \left((m+1)^2 - \delta^2 \right) \tilde{\psi}_{m+1} \quad (3.22a)$$

$$\hat{J}_- \tilde{\psi}_m = \tilde{\psi}_{m-1} \quad (3.22b)$$

$$\hat{J}_0 \tilde{\psi}_m = \left(m + \frac{1}{2} \right) \tilde{\psi}_m . \quad (3.22c)$$

The abstract Hilbert spaces fall again into three cases. There are counterparts of \mathcal{H}_{HH}^\pm , with the sign in the spectrum of \hat{J}_0 inverted, and there is a continuum of Hilbert spaces isomorphic to $\mathcal{H}_{HH}^\epsilon$. Thus, the spectrum of \hat{J}_0 is unbounded below in *all* the Hilbert spaces obtained in the doubly antiholomorphic representation, and in addition bounded above in the counterparts of \mathcal{H}_{HH}^\pm . This is a drastic violation of the classical range of J_0 . Again, the vectors in the counterparts of \mathcal{H}_{HH}^\pm can be represented by antianalytic functions, whereas the spaces isomorphic to $\mathcal{H}_{HH}^\epsilon$ all contain vectors that cannot be represented in this way.

Taken together, the set of Hilbert spaces obtained in the doubly holomorphic representation and the doubly antiholomorphic representation contains precisely those Hilbert spaces that were obtained in Refs. [3,22] in the generalized angular momentum eigenstate representation

$$\hat{J}_0 |m\rangle = \left(m + \frac{1}{2} \right) |m\rangle \quad (3.23a)$$

$$\hat{J}_+ |m\rangle = \sqrt{(m+1)^2 - \delta^2} |m+1\rangle \quad (3.23b)$$

$$\hat{J}_- |m\rangle = \sqrt{m^2 - \delta^2} |m-1\rangle , \quad (3.23c)$$

which treats the raising and lowering operators in a manifestly symmetric fashion.

It is natural to ask what happens in the mixed representations where one of the oscillators is holomorphic and the other antiholomorphic. This will be explored in section IV.

B. Doubly holomorphic path-integral quantization

We wish now to write a path integral for a “propagation amplitude” with boundary conditions such that the amplitude could be related to the Dirac quantization of subsection III A. For definiteness, we only consider the doubly holomorphic representation. The results for the doubly antiholomorphic representation are analogous.

Our starting point is the formal path integral expression

$$G(\alpha_I; \bar{\beta}_I) = \int_{\bar{z}_I(t_1)=\bar{\beta}_I}^{z_I(t_2)=\alpha_I} \mathcal{D}(z_I, \bar{z}_I, N) \exp(iS) \quad (3.24)$$

where the action is given by

$$S = \int_{t_1}^{t_2} dt (-i\bar{z}_I \dot{z}_I - N\mathcal{H}) - iz_I(t_1)\bar{z}_I(t_1) \quad (3.25)$$

A sum over the repeated index I in Eq. (3.25) is understood. The quantities to be integrated over are $z_I(t)$, $\bar{z}_I(t)$ and $N(t)$, with the final values of z_I fixed to α_I , and the initial values of \bar{z}_I fixed to $\bar{\beta}_I$. The action (3.25) is the appropriate one for the classical boundary value problem with these boundary conditions, provided z_I and \bar{z}_I are regarded as independent quantities and not each others’ complex conjugates. Through integrations by parts, this action can be written in a more symmetric fashion with respect to the barred and unbarred variables [24].

We wish now to give a meaning to this formal expression.

First, the action (3.25) has a gauge symmetry corresponding to reparametrizations in t , and this gauge freedom must be eliminated in the path integral. Adopting the proper-time gauge $\dot{N} = 0$ [8,10], we obtain

$$G(\alpha_I; \bar{\beta}_I) = (t_2 - t_1) \int_{\bar{z}_I(t_1)=\bar{\beta}_I}^{z_I(t_2)=\alpha_I} dN \int \mathcal{D}(z_I, \bar{z}_I) \exp(iS) \quad (3.26)$$

where S is still given by Eq. (3.25) but N is now just a number.

Next, consider the $\mathcal{D}(z_I, \bar{z}_I)$ integrals. For N real, the integrand and the boundary conditions are in essence identical to those in the path integral which would give the conventional coherent state matrix elements of the time evolution operator for a system of two unconstrained harmonic oscillators [24,25]. We define the measure $\mathcal{D}(z_I, \bar{z}_I)$ to be equal to the path measure for two such unconstrained oscillators, multiplied for later convenience by the numerical factor $1/(2\pi)$. The result is

$$G(\alpha_I; \bar{\beta}_I) = \frac{1}{2\pi} \int dT \exp(\alpha_1 \bar{\beta}_1 e^{-iT} + \alpha_2 \bar{\beta}_2 e^{iT} + 2i\delta T) \quad (3.27)$$

where we have written $(t_2 - t_1)N = T$. We take the integrand in Eq. (3.27) to be valid also when analytically continued to complex values of T , and we seek to define the remaining integral over T as a contour integral in the complex T plane.

Suppose first that α_I and $\bar{\beta}_I$ are all real and positive. Define in the complex T plane a contour \mathcal{C}_1 consisting of three straight lines: a line parallel to the imaginary axis from $-\pi + i\infty$ to $-\pi$, a line along the real axis from $-\pi$ to π , and a line parallel to the imaginary axis from π to $\pi + i\infty$. Similarly, define a contour \mathcal{C}_2 to be the mirror image of \mathcal{C}_1 with respect to the real axis, going first from $-\pi - i\infty$ to $-\pi$, then from $-\pi$ to π , and finally from π to $\pi - i\infty$. The integral in Eq. (3.27) along each of these contours converges, with the result [35]

$$\left(\frac{\alpha_1\bar{\beta}_1}{\alpha_2\bar{\beta}_2}\right)^\delta I_{\pm 2\delta} \left(2\sqrt{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}\right) \quad (3.28)$$

where $I_{\pm 2\delta}$ is the modified Bessel function of the first kind, and the upper and lower signs refer respectively to \mathcal{C}_1 and \mathcal{C}_2 . If 2δ is an integer, the contributions from the lines parallel to the imaginary axis cancel, and both \mathcal{C}_1 and \mathcal{C}_2 are equivalent to the contour along the real axis from $-\pi$ to π .

The central observation is now that the expression (3.28) defines, when analytically continued to complex values of α_I and $\bar{\beta}_I$, quantities that have an interpretation in terms of the Hilbert spaces $\mathcal{H}_{\text{HH}}^\pm$. We shall first exhibit this interpretation, and then discuss in what sense such objects can be seen as arising from the path integral.

Define the amplitudes G_\pm as

$$G_\pm(\alpha_I; \bar{\beta}_I) = \left(\frac{\alpha_1\bar{\beta}_1}{\alpha_2\bar{\beta}_2}\right)^\delta I_{\pm 2|\delta|} \left(2\sqrt{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}\right) \quad , \quad (3.29)$$

where G_+ is defined for all δ , and G_- is defined only in the case $0 < |\delta| < \frac{1}{2}$. The arguments α_I and $\bar{\beta}_I$ are complex-valued, and G_\pm is understood as a function on an appropriate Riemann sheet in each of the arguments. We shall show that G_\pm are the matrix elements of the identity operator in $\mathcal{H}_{\text{HH}}^\pm$ in a generalized coherent state basis.

First, expanding the Bessel function in Eq. (3.29) as a power series and using (3.15), one obtains

$$G_\pm(\alpha_I; \bar{\beta}_I) = \sum_m \frac{\psi_m(\alpha_I) \overline{\psi_m(\bar{\beta}_I)}}{(\psi_m, \psi_m)} \quad (3.30)$$

where the values of the summation index m and the inner product (ψ_m, ψ_m) refer to $\mathcal{H}_{\text{HH}}^\pm$. Identifying $\{\overline{\psi_m}\}$ as a basis of the dual space of $\mathcal{H}_{\text{HH}}^\pm$, one recognizes Eq. (3.30) as the resolution of the identity operator in $\mathcal{H}_{\text{HH}}^\pm$ with respect to the basis $\{\psi_m\}$. The amplitude G_\pm therefore defines the identity operator in $\mathcal{H}_{\text{HH}}^\pm$.

To see the relation to coherent states, define for fixed $\bar{\beta}_I$ the functions $\Phi_{\bar{\beta}_I}^\pm$ by

$$\Phi_{\bar{\beta}_I}^\pm(z_I) = G_\pm(z_I; \bar{\beta}_I) \quad . \quad (3.31)$$

Here $\Phi_{\bar{\beta}_I}^+$ is defined for all $\bar{\beta}_I$ and all δ , whereas $\Phi_{\bar{\beta}_I}^-$ is defined only for $\bar{\beta}_1 \neq 0$ when $-\frac{1}{2} < \delta < 0$, and only for $\bar{\beta}_2 \neq 0$ when $0 < \delta < \frac{1}{2}$. From Eq. (3.30) one readily sees that the functions $\Phi_{\bar{\beta}_I}^\pm$ define vectors in $\mathcal{H}_{\text{HH}}^\pm$, and further that $(\Phi_{\bar{\beta}_I}^\pm, \psi) = \psi(\bar{\beta}_I)$ for any $\psi \in \mathcal{H}_{\text{HH}}^\pm$. In particular,

$$\left(\Phi_{\alpha_I}^{\pm}, \Phi_{\bar{\beta}_I}^{\pm}\right) = G_{\pm}(\alpha_I; \bar{\beta}_I) \quad . \quad (3.32)$$

Thus, $G_{\pm}(\alpha_I; \bar{\beta}_I)$ are the matrix elements of the identity operator in $\mathcal{H}_{\text{HH}}^{\pm}$ with respect to the states $\{\Phi_{\bar{\beta}_I}^{\pm}\}$.

The states $\Phi_{\bar{\beta}_I}^{\pm}$ satisfy

$$\hat{J}_- \Phi_{\bar{\beta}_I}^{\pm} = \bar{\beta}_1 \bar{\beta}_2 \Phi_{\bar{\beta}_I}^{\pm} \quad , \quad (3.33)$$

by which they are known as a system of annihilation operator coherent states associated with the $SO(2, 1)$ algebra [26]. They are analogous to conserved charge coherent states, and $\Phi_{\bar{\beta}_I}^+$ in fact reduces to a conserved charge coherent state when 2δ is an integer [26,36]. In particular, $\{\Phi_{\bar{\beta}_I}^{\pm}\}$ form an overcomplete set in $\mathcal{H}_{\text{HH}}^{\pm}$. The resolution of the identity is most conveniently written by noticing that apart from a normalization factor, $\Phi_{\bar{\beta}_I}^{\pm}$ depend on $\bar{\beta}_1$ and $\bar{\beta}_2$ only through the product $\bar{\beta}_1 \bar{\beta}_2$. One can therefore define (adopting for the moment Dirac's bra-ket notation) the states $|\bar{\lambda}, \pm\rangle \in \mathcal{H}_{\text{HH}}^{\pm}$, labeled by a single complex number $\bar{\lambda}$, by

$$\Phi_{\bar{\beta}_I}^{\pm} = \left(\frac{\bar{\beta}_1}{\bar{\beta}_2}\right)^{\delta} |\bar{\beta}_1 \bar{\beta}_2, \pm\rangle \quad , \quad (3.34)$$

where $|\bar{\lambda}, +\rangle$ is defined for all $\bar{\lambda}$, and $|\bar{\lambda}, -\rangle$ is defined for $\bar{\lambda} \neq 0$. Using Eq. (3.19), one then obtains

$$\hat{\mathbb{1}} = \int \frac{d\bar{\lambda} \wedge d\lambda}{\pi i} K_{2\delta} \left(2\sqrt{\lambda \bar{\lambda}}\right) |\bar{\lambda}, \pm\rangle \langle \lambda, \pm| \quad . \quad (3.35)$$

The states $\{\Phi_{\bar{\beta}_I}^{\pm}\}$ form also a system of minimum uncertainty coherent states [26]. To see this, define the Hermitian operators

$$\begin{aligned} \hat{J}_x &= \frac{1}{2} (\hat{J}_+ + \hat{J}_-) \\ \hat{J}_y &= \frac{1}{2i} (\hat{J}_+ - \hat{J}_-) \quad , \end{aligned} \quad (3.36)$$

in terms of which the commutators take the standard form of the $SO(2, 1)$ algebra,

$$[\hat{J}_x, \hat{J}_y] = -i\hat{J}_0 \quad (3.37a)$$

$$[\hat{J}_y, \hat{J}_0] = i\hat{J}_x \quad (3.37b)$$

$$[\hat{J}_0, \hat{J}_x] = i\hat{J}_y \quad . \quad (3.37c)$$

From (3.37a), the Heisenberg uncertainty relation for \hat{J}_x and \hat{J}_y is [26]

$$\langle (\Delta \hat{J}_x)^2 \rangle \langle (\Delta \hat{J}_y)^2 \rangle \geq \frac{1}{4} \langle \hat{J}_0 \rangle^2 \quad . \quad (3.38)$$

In the state $\Phi_{\bar{\beta}_I}^{\pm}$, one has by Eq. (3.33)

$$\begin{aligned}\langle \hat{J}_x \rangle &= \text{Re}(\beta_1 \beta_2) \\ \langle \hat{J}_y \rangle &= \text{Im}(\beta_1 \beta_2) \quad ,\end{aligned}\tag{3.39}$$

and a short computation shows that the uncertainty relation (3.38) holds with the equality sign. The interpretation is that the state $\Phi_{\beta_I}^\pm$ is as closely as possible peaked around the classical solution for which $J_+ = \beta_1 \beta_2$, $J_- = \bar{\beta}_1 \bar{\beta}_2$.

Finally, let us return to the question of recovering the amplitudes G_\pm from the path integral for general complex values of the arguments. Recall that we first defined G_\pm only for positive values of the arguments by giving a contour prescription in Eq. (3.27), and we then analytically continued to complex arguments. It is of interest to ask whether there is a contour in Eq. (3.27) that would yield G_\pm directly for all complex values of the arguments. If 2δ is an integer (in which case only G_+ is defined) the answer is yes, and the contour can be chosen to be along the real T axis from $-\pi$ to π . For 2δ not an integer, however, the answer is negative. For any contour such that Eq. (3.27) converges for all α_I and $\bar{\beta}_I$, the resulting G is a single-valued analytic function in its arguments, and thus cannot be equal to G_\pm .

It is possible to obtain G_\pm for general δ and complex arguments living on the appropriate Riemann sheets if one allows the contour in Eq. (3.27) to depend on the arguments. This can be achieved by contours consisting of two semi-infinite vertical pieces joined by a piece along the real axis, as in \mathcal{C}_1 and \mathcal{C}_2 , but now choosing the locations of the vertical pieces to depend on the arguments of G_\pm in a suitable way. It is however not clear in what sense such a contour prescription could be justified in terms of the original path integral (3.24).

IV. QUANTUM THEORY IN THE HOLOMORPHIC-ANTIHOLOMORPHIC REPRESENTATION

In this section we shall repeat the analysis of section III, but now choosing a mixed representation where the wave functions are analytic in one oscillator and antianalytic in the other. Subsection IV A carries out the Dirac quantization, and subsection IV B will construct a propagation amplitude from a path integral.

A. Holomorphic-antiholomorphic Dirac quantization

Our starting point is again the operator algebra \mathcal{A} defined in subsection III A. We now represent \mathcal{A} on the vector space

$$V_{\text{HA}} = \{\chi(z_1, \bar{z}_2) \mid \chi \text{ analytic in } z_1, \text{ antianalytic in } \bar{z}_2\}\tag{4.1}$$

by

$$\begin{aligned}\hat{z}_1 \chi &= z_1 \chi \\ \hat{\bar{z}}_1 \chi &= \frac{\partial \chi}{\partial z_1} \\ \hat{z}_2 \chi &= -\frac{\partial \chi}{\partial \bar{z}_2} \\ \hat{\bar{z}}_2 \chi &= \bar{z}_2 \chi \quad .\end{aligned}\tag{4.2}$$

We refer to this as the holomorphic-antiholomorphic representation. With the quantum constraint $\hat{\mathcal{H}}$ still given by Eq. (3.4), the space of solutions to the constraint equation $\hat{\mathcal{H}}\chi = 0$ is

$$V_{\text{HA},0} = \left\{ \chi \in V_{\text{HA}} \mid \chi(z_1, \bar{z}_2) = (z_1 \bar{z}_2)^{\delta-(1/2)} \rho(z_1/\bar{z}_2) \right\} \quad , \quad (4.3)$$

where ρ is a complex analytic function of its argument. Again, we defer the question of the singularities in ρ until later.

The star-algebra $\mathcal{A}_{\text{phy}}^{(*)}$ of physical operators is as in section III. Writing a vector in $V_{\text{HA},0}$ as in Eq. (4.3), the action of the \hat{J} 's is

$$(\hat{J}_+ \rho)(w) = -\left(\delta - \frac{1}{2}\right) w \rho(w) + w^2 \rho'(w) \quad (4.4a)$$

$$(\hat{J}_- \rho)(w) = \frac{\left(\delta - \frac{1}{2}\right)}{w} \rho(w) + \rho'(w) \quad (4.4b)$$

$$(\hat{J}_0 \rho)(w) = w \rho'(w) \quad , \quad (4.4c)$$

where the prime denotes derivative with respect to the argument. This representation is again highly reducible. Consider therefore the subspace where ρ is a linear combination of arbitrary powers,

$$\tilde{V}_{\text{HA},0} = \text{Span} \left\{ \chi_m \in V_{\text{HA},0} \mid \chi_m(z_1, \bar{z}_2) = (z_1 \bar{z}_2)^{\delta-(1/2)} (z_1/\bar{z}_2)^{m+(1/2)}, m \in \mathbf{C} \right\} \quad . \quad (4.5)$$

The action of the \hat{J} 's on the basis vectors χ_m is

$$\hat{J}_+ \chi_m = (m+1-\delta) \chi_{m+1} \quad (4.6a)$$

$$\hat{J}_- \chi_m = (m+\delta) \chi_{m-1} \quad (4.6b)$$

$$\hat{J}_0 \chi_m = \left(m + \frac{1}{2}\right) \chi_m \quad , \quad (4.6c)$$

and $\tilde{V}_{\text{HA},0}$ therefore carries a representation of $\mathcal{A}_{\text{phy}}^{(*)}$. This representation is still reducible, but the subspaces carrying irreducible representations are straightforward to find. All these irreducible representations are cyclic, with some $\chi_m \in \tilde{V}_{\text{HA},0}$ as the cyclic vector. Representations with countably infinite dimension exist for any δ , whereas finite dimensional representations exist only when 2δ is a positive integer.

We seek those irreducible representations that admit an inner product satisfying the Hermitian conjugacy relations (3.13). Again, such an inner product exists only for a subset of the irreducible representations, but when it exists, it is unique (up to an overall constant). There are just three different cases.

For $\delta < \frac{1}{2}$, one admissible vector space is

$$V_{\text{HA}}^1 = \text{Span} \left\{ \chi_m \in \tilde{V}_{\text{HA},0} \mid m = -\delta + k, k = 0, 1, 2, \dots \right\} \quad . \quad (4.7)$$

The inner product is

$$(\chi_m, \chi_{m'}) = \frac{\Gamma(m+1+\delta)}{\Gamma(m+1-\delta)} \delta_{m,m'} \quad , \quad (4.8)$$

where $\delta_{m,m'}$ stands for the Kronecker delta. For $\delta \leq 0$, the representation of $\mathcal{A}_{\text{phy}}^{(*)}$ on V_{HA}^1 is isomorphic to that on V_{HH}^+ in subsection III A, and for $0 < \delta < \frac{1}{2}$ it is isomorphic to that on V_{HH}^- . In terms of the basis vectors, the isomorphism is

$$\chi_{-\delta+k} \longmapsto \frac{1}{\Gamma(k+1-2\delta)} \psi_{-\delta+k} \quad . \quad (4.9)$$

The eigenenergies of the first oscillator are now the positive half-integers $k + \frac{1}{2}$, and those of the second oscillator are respectively $k + \frac{1}{2} - 2\delta$.

For $\delta < \frac{1}{2}$, a second admissible vector space is

$$V_{\text{HA}}^2 = \text{Span} \left\{ \chi_m \in \tilde{V}_{\text{HA},0} \mid m = \delta - 1 - k, k = 0, 1, 2, \dots \right\} \quad . \quad (4.10)$$

The inner product is

$$(\chi_m, \chi_{m'}) = \frac{\Gamma(\delta - m)}{\Gamma(-\delta - m)} \delta_{m,m'} \quad . \quad (4.11)$$

The eigenenergies of the second oscillator are $-k - \frac{1}{2}$, and those of the first oscillator respectively $2\delta - k - \frac{1}{2}$. In particular, the spectrum of \hat{J}_0 is bounded above by $\delta - \frac{1}{2}$, and thus negative definite. The representation of $\mathcal{A}_{\text{phy}}^{(*)}$ on V_{HA}^2 is therefore not isomorphic to any of the representations obtained in the doubly holomorphic representation. Instead, it is isomorphic to a representation on the doubly antiholomorphic counterpart of V_{HH}^+ or V_{HH}^- , depending on the value of δ , as was discussed at the end of subsection III A.

Finally, if $|\delta| < \frac{1}{2}$, there exists a one-parameter family of admissible vector spaces, labeled by a real number ϵ satisfying $|\delta| < \epsilon < 1 - |\delta|$. These spaces are

$$V_{\text{HA}}^\epsilon = \text{Span} \left\{ \chi_m \in \tilde{V}_{\text{HA},0} \mid m = \epsilon + k, k \in \mathbf{Z} \right\} \quad , \quad (4.12)$$

and the inner product is given by Eq. (4.8). These representations are isomorphic to those obtained in subsection III A on V_{HH}^ϵ , with the isomorphism

$$\chi_m \longmapsto \frac{1}{\Gamma(m+1-\delta)} \psi_m \quad . \quad (4.13)$$

These three cases give the complete list of the inner product spaces. The respective abstract Hilbert spaces $\mathcal{H}_{\text{HA}}^1$, $\mathcal{H}_{\text{HA}}^2$ and $\mathcal{H}_{\text{HA}}^\epsilon$ are obtained by Cauchy completion. Vectors in $\mathcal{H}_{\text{HA}}^1$ and $\mathcal{H}_{\text{HA}}^2$ can be represented by complex analytic/antianalytic functions of the form given in Eq. (4.3). In $\mathcal{H}_{\text{HA}}^1$, $w^{\delta-(1/2)}\rho(w)$ is analytic in the open unit disk $|w| < 1$. In $\mathcal{H}_{\text{HA}}^2$, $w^{(1/2)-\delta}\rho(w)$ is analytic in the domain $|w| > 1$ in the compactified complex plane, including the point $w = \infty$. The spaces $\mathcal{H}_{\text{HA}}^\epsilon$, on the other hand, all contain vectors that cannot be represented by functions of the form given in Eq. (4.3), the reason being the divergence of the infinite series expression for $\rho(w)$ for $|w| \neq 1$.

Therefore, if we require that all vectors in the Hilbert space be representable by functions of the form given in Eq. (4.3), only the spaces $\mathcal{H}_{\text{HA}}^1$ and $\mathcal{H}_{\text{HA}}^2$ remain. For $\delta < 0$, the inner products in these two spaces can be expressed as holomorphic integrals. A formula valid for any $\chi_{(1)}, \chi_{(2)} \in \mathcal{H}_{\text{HA}}^1$ is

$$(\chi_{(1)}, \chi_{(2)}) = \frac{1}{\Gamma(-2\delta)} \int_{|w|<1} \frac{d\bar{w} \wedge dw}{2\pi i} (w\bar{w})^{\delta-(1/2)} (1 - w\bar{w})^{-2\delta-1} \overline{\rho_{(1)}(w)} \rho_{(2)}(w) \quad . \quad (4.14)$$

For $\mathcal{H}_{\text{HA}}^2$ an analogous formula holds with the integration domain $|w| > 1$.

We have thus recovered a set of Hilbert spaces carrying representations of $\mathcal{A}_{\text{phy}}^{(*)}$. From the discussion at the end of subsection III A we see that this set does not contain all the Hilbert spaces that were obtained in Refs. [3,22] in the symmetric representation (3.23). The reason is that the choice of the holomorphic-antiholomorphic representation has introduced an asymmetry, this time between the two oscillators: the representation (4.6) is not invariant under reversing the sign of δ . In particular, the holomorphic-antiholomorphic representation leads to Hilbert spaces only for $\delta < \frac{1}{2}$.

Suppose that, instead of the holomorphic-antiholomorphic representation, one starts with the antiholomorphic-holomorphic representation where the two oscillators in Eqs. (4.1) and (4.2) are interchanged. The analysis proceeds in exact parallel to the above. The counterpart of Eqs. (4.6) will have the inverse sign in δ , and one arrives at counterparts of $\mathcal{H}_{\text{HA}}^1$ and $\mathcal{H}_{\text{HA}}^2$ where the oscillators are interchanged, as well as at a set of Hilbert spaces isomorphic to $\mathcal{H}_{\text{HA}}^\epsilon$. Taken together, the two mixed representations therefore recover all the Hilbert spaces that were obtained in the representation (3.23).

B. Holomorphic-antiholomorphic path-integral quantization

We wish to write a path integral which could be related to the Hilbert spaces obtained in subsection IV A. For definiteness, we only consider the holomorphic-antiholomorphic representation, and we assume $\delta < \frac{1}{2}$ throughout the subsection. Analogous results hold in the case $\delta > -\frac{1}{2}$ for the antiholomorphic-holomorphic representation.

Our starting point is now the expression

$$G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2) = \int_{\substack{z_1(t_2)=\alpha_1 \\ \bar{z}_2(t_2)=\bar{\alpha}_2 \\ \bar{z}_1(t_1)=\bar{\beta}_1 \\ z_2(t_1)=\beta_2}} \mathcal{D}(z_I, \bar{z}_I, N) \exp(iS) \quad (4.15)$$

with the action

$$S = \int_{t_1}^{t_2} dt (-i\bar{z}_I \dot{z}_I - N\mathcal{H}) - iz_1(t_1)\bar{z}_1(t_1) + iz_2(t_2)\bar{z}_2(t_2) \quad . \quad (4.16)$$

A sum over the repeated index I in Eq. (4.16) is understood. The boundary conditions in the integral consist of fixing the final values of z_1 and \bar{z}_2 respectively to α_1 and $\bar{\alpha}_2$, and the initial values of \bar{z}_1 and z_2 respectively to $\bar{\beta}_1$ and β_2 . The action (4.16) is the appropriate one for the classical boundary value problem with these boundary conditions, provided z_I and \bar{z}_I are considered independent quantities.

To give a meaning to this formal expression, we again first adopt the proper time gauge $\dot{N} = 0$. The result is

$$G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2) = (t_2 - t_1) \int dN \int_{\substack{z_1(t_2)=\alpha_1 \\ \bar{z}_2(t_2)=\bar{\alpha}_2 \\ \bar{z}_1(t_1)=\bar{\beta}_1 \\ z_2(t_1)=\beta_2}} \mathcal{D}(z_I, \bar{z}_I) \exp(iS) \quad . \quad (4.17)$$

Consider then the $\mathcal{D}(z_I, \bar{z}_I)$ integrals. The path integral for the first oscillator is formally identical to that in section III. The path integral for the second oscillator is formally closely similar. However, to define the integral in a consistent manner, we now need a more careful discussion of the factor ordering.

Recall that for a single unconstrained oscillator, the usual holomorphic path integral gives just the exponential of the classical action with the appropriate boundary data [24]. More precisely, the classical action is evaluated with the Hamiltonian $z\bar{z}$, and the result is the coherent state matrix element of the time evolution operator for the normal ordered quantum Hamiltonian $\hat{z}\hat{\bar{z}}$. In other words, the classical Hamiltonian $z\bar{z}$ corresponds to the quantum Hamiltonian in which the operator represented by differentiation gets ordered to the right. This was the definition adopted in subsection III B for both oscillators, and we shall again adopt this definition here for the first oscillator. For the second oscillator, on the other hand, we would like to define the path integral in Eq. (4.17) so as to correspond to the operator ordering $-\hat{z}_2\hat{\bar{z}}_2$, in which the operator represented by differentiation is now on the left. This suggests that the path integral for the second oscillator should again be given by the exponential of the classical action, but now evaluated with the classical Hamiltonian $1 - z_2\bar{z}_2$ rather than just $-z_2\bar{z}_2$. We shall adopt this definition. It will be shown that the resulting amplitude can be interpreted in terms of the spaces $\mathcal{H}_{\text{HA}}^1$ and $\mathcal{H}_{\text{HA}}^2$ of subsection IV A.

With the above definitions, the amplitude takes the form

$$G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2) = -i \int dT \exp \left[(\alpha_1 \bar{\beta}_1 - \bar{\alpha}_2 \beta_2) e^{-iT} + i(2\delta - 1)T \right] \quad . \quad (4.18)$$

We have written $(t_2 - t_1)N = T$, and the integrand is taken to be valid in all of the complex T plane. For convenience, the measure has been chosen to include a numerical factor of $-i$.

The remaining integral over T is defined via complex integration and analytic continuation in the same fashion as in subsection III B. For example, in the case $\bar{\alpha}_2\beta_2 - \alpha_1\bar{\beta}_1 > 0$ the T contour can be taken along the positive imaginary axis. The result is

$$G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2) = \frac{\Gamma(1 - 2\delta)}{(\bar{\alpha}_2\beta_2 - \alpha_1\bar{\beta}_1)^{1-2\delta}} \quad . \quad (4.19)$$

We understand $G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2)$ as a complex analytic function on an appropriate Riemann sheet in each of its arguments. We shall show that $G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2)$ are the matrix elements of the identity operator in $\mathcal{H}_{\text{HA}}^1$ and $\mathcal{H}_{\text{HA}}^2$ in a generalized coherent state basis which differs from the one discussed in subsection III B.

Suppose first that the inequalities

$$|\alpha_1/\bar{\alpha}_2| < 1 \quad (4.20a)$$

$$|\bar{\beta}_1/\beta_2| < 1 \quad (4.20b)$$

hold. Expanding G in the quantity $\alpha_1 \bar{\beta}_1 / \bar{\alpha}_2 \beta_2$ by the binomial theorem and using Eq. (4.8), one has

$$G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2) = \sum_m \frac{\chi_m(\alpha_1, \bar{\alpha}_2) \overline{\chi_m(\beta_1, \bar{\beta}_2)}}{(\chi_m, \chi_m)} , \quad (4.21)$$

where the values of the summation index m and the inner product (χ_m, χ_m) refer to $\mathcal{H}_{\text{HA}}^1$. Identifying $\{\bar{\chi}_m\}$ as a basis of the dual space of $\mathcal{H}_{\text{HA}}^1$, one recognizes Eq. (4.21) as the resolution of the identity operator in $\mathcal{H}_{\text{HA}}^1$ with respect to the basis $\{\chi_m\}$. The amplitude G therefore defines the identity operator in $\mathcal{H}_{\text{HA}}^1$.

To see the relation to coherent states, still assuming Eqs. (4.20) to hold (so that in particular $\beta_2 \neq 0$), define for fixed $\bar{\beta}_1$ and β_2 the function $\Psi_{\bar{\beta}_1, \beta_2}$ by

$$\Psi_{\bar{\beta}_1, \beta_2}(z_1, \bar{z}_2) = G(z_1, \bar{z}_2; \bar{\beta}_1, \beta_2) . \quad (4.22)$$

From Eq. (4.21) one sees that $\Psi_{\bar{\beta}_1, \beta_2}$ defines a vector in $\mathcal{H}_{\text{HA}}^1$. (Note that this would not hold if the condition (4.20b) were relaxed.) Further, $(\Psi_{\bar{\beta}_1, \beta_2}, \chi) = \chi(\beta_1, \bar{\beta}_2)$ for any $\chi \in \mathcal{H}_{\text{HA}}^1$. In particular,

$$(\Psi_{\bar{\alpha}_1, \alpha_2}, \Psi_{\bar{\beta}_1, \beta_2}) = G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2) . \quad (4.23)$$

Thus, $G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2)$ are the matrix elements of the identity operator in $\mathcal{H}_{\text{HA}}^1$ with respect to the states $\{\Psi_{\bar{\beta}_1, \beta_2}\}$.

A direct computation shows that $\Psi_{\bar{\beta}_1, \beta_2}$ can be obtained by operating on the state $\chi_{-\delta}$ annihilated by \hat{J}_- with the exponential of the raising operator,

$$\Psi_{\bar{\beta}_1, \beta_2} = \Gamma(1 - 2\delta)(\beta_2)^{2\delta-1} \exp\left[\left(\bar{\beta}_1/\beta_2\right) \hat{J}_+\right] \chi_{-\delta} . \quad (4.24)$$

This means that $\{\Psi_{\bar{\beta}_1, \beta_2}\}$ are a system of displacement operator coherent states associated with the $SO(2, 1)$ algebra [26,27], and from the construction it is clear that they are an overcomplete set in $\mathcal{H}_{\text{HA}}^1$. For $\delta < 0$, one can use Eq. (4.14) to write the resolution of the identity in terms of the states $\{\Psi_{\bar{\beta}_1, \beta_2}\}$ in a form analogous to Eq. (3.35).

Note that $\Psi_{\bar{\beta}_1, \beta_2}$ is a lowering operator coherent state only in the trivial case $\bar{\beta}_1 = 0$. Also, it can be verified that $\Psi_{\bar{\beta}_1, \beta_2}$ is a minimum uncertainty coherent state for \hat{J}_x and \hat{J}_y only when $\bar{\beta}_1/\beta_2$ is either purely real or purely imaginary.

Suppose then that the inequalities (4.20) are reversed. In this case it is seen in an analogous fashion that G defines, up to a phase $(-1)^{2\delta-1}$, the identity operator in $\mathcal{H}_{\text{HA}}^2$. The functions $\Psi_{\bar{\beta}_1, \beta_2}$ (4.22) again define displacement operator coherent states in $\mathcal{H}_{\text{HA}}^2$, but now with respect to the lowering operator,

$$\Psi_{\bar{\beta}_1, \beta_2} = \Gamma(1 - 2\delta)(-\bar{\beta}_1)^{2\delta-1} \exp\left[-\left(\beta_2/\bar{\beta}_1\right) \hat{J}_-\right] \chi_{\delta-1} . \quad (4.25)$$

Again, $G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2)$ are (up to the phase $(-1)^{2\delta-1}$) the matrix elements of the identity operator in $\mathcal{H}_{\text{HA}}^2$ with respect to the states $\{\Psi_{\bar{\beta}_1, \beta_2}\}$.

V. DIRAC QUANTIZATION IN THE POSITION REPRESENTATION

In this section we shall carry out the Dirac quantization in a position representation in which the state vectors are functions of x_I . The steps are closely similar to those in the previous sections.

We now represent the operator algebra \mathcal{A} on the space of smooth functions of two real variables,

$$V_{\text{pos}} = \{\xi(x_1, x_2) \mid \xi \in C^\infty(\mathbf{R}^2)\} \quad . \quad (5.1)$$

\hat{z}_I and $\hat{\bar{z}}_I$ are taken to act as the usual creation and annihilation operators in the position representation of unconstrained oscillators,

$$\begin{aligned} \hat{z}_I \xi &= \frac{1}{\sqrt{2}} \left(x_I - \frac{\partial}{\partial x_I} \right) \xi \\ \hat{\bar{z}}_I \xi &= \frac{1}{\sqrt{2}} \left(x_I + \frac{\partial}{\partial x_I} \right) \xi \quad . \end{aligned} \quad (5.2)$$

With the Hamiltonian constraint $\hat{\mathcal{H}}$ (3.4), the space of all solutions to the constraint equation $\hat{\mathcal{H}}\xi = 0$ does not have a simple characterization as in the previous sections. We therefore immediately restrict the attention to the space spanned by solutions that are separable in x_1 and x_2 . This space, denoted by $\tilde{V}_{\text{pos},0}$, can be expressed as a direct sum of four subspaces,

$$\tilde{V}_{\text{pos},0} = V_{UU} \oplus V_{UV} \oplus V_{VU} \oplus V_{VV} \quad , \quad (5.3)$$

such that

$$\begin{aligned} V_{UU} &= \text{Span} \{ \xi_{UU,m} \in V_{\text{pos}} \mid m \in \mathbf{C} \} \\ V_{UV} &= \text{Span} \{ \xi_{UV,m} \in V_{\text{pos}} \mid m \in \mathbf{C} \} \\ V_{VU} &= \text{Span} \{ \xi_{VU,m} \in V_{\text{pos}} \mid m \in \mathbf{C} \} \\ V_{VV} &= \text{Span} \{ \xi_{VV,m} \in V_{\text{pos}} \mid m \in \mathbf{C} \} \quad , \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} \xi_{UU,m}(x_1, x_2) &= U\left(-\delta - m - \frac{1}{2}, \sqrt{2}x_1\right) U\left(\delta - m - \frac{1}{2}, \sqrt{2}x_2\right) \\ \xi_{UV,m}(x_1, x_2) &= U\left(-\delta - m - \frac{1}{2}, \sqrt{2}x_1\right) V\left(\delta - m - \frac{1}{2}, \sqrt{2}x_2\right) \\ \xi_{VU,m}(x_1, x_2) &= V\left(-\delta - m - \frac{1}{2}, \sqrt{2}x_1\right) U\left(\delta - m - \frac{1}{2}, \sqrt{2}x_2\right) \\ \xi_{VV,m}(x_1, x_2) &= V\left(-\delta - m - \frac{1}{2}, \sqrt{2}x_1\right) V\left(\delta - m - \frac{1}{2}, \sqrt{2}x_2\right) \quad . \end{aligned} \quad (5.5)$$

Here U and V are the two independent solutions to the parabolic cylinder equation. Their properties, including the behavior under the action of \hat{z}_I and $\hat{\bar{z}}_I$ (5.2), can be found in Ref. [37].

All the four spaces in Eqs. (5.4) are invariant under the action of $\mathcal{A}_{\text{phy}}^{(*)}$. The representation of $\mathcal{A}_{\text{phy}}^{(*)}$ on V_{UU} is isomorphic to that on $\tilde{V}_{\text{HH},0}$ (3.11) encountered in section III, with

the isomorphism $\xi_{UU,m} \mapsto \psi_m$. The representation of $\mathcal{A}_{\text{phy}}^{(*)}$ on V_{UV} is isomorphic to that on $\tilde{V}_{\text{HA},0}$ (4.5) encountered in section IV, with the isomorphism $\xi_{UV,m} \mapsto \chi_m$. Similarly, the representation on V_{VV} is isomorphic to that given by Eqs. (3.22) on the doubly antiholomorphic counterpart of $\tilde{V}_{\text{HH},0}$, and the representation on V_{VU} is isomorphic to that on the antiholomorphic-holomorphic counterpart of $\tilde{V}_{\text{HA},0}$.

We therefore see that all the irreducible representations of $\mathcal{A}_{\text{phy}}^{(*)}$ that were obtained in the doubly holomorphic, doubly antiholomorphic, and mixed representations are isomorphic to representations of $\mathcal{A}_{\text{phy}}^{(*)}$ on linear subspaces of $\tilde{V}_{\text{pos},0}$. Conversely, all irreducible representations of $\mathcal{A}_{\text{phy}}^{(*)}$ on linear subspaces of $\tilde{V}_{\text{pos},0}$ are isomorphic to representations that were obtained in the doubly holomorphic, doubly antiholomorphic, or mixed representations. Thus, the imposition of the Hermitian conjugacy relations (3.13) for an inner product on subspaces of $\tilde{V}_{\text{pos},0}$ proceeds exactly as before, and the resulting abstract Hilbert spaces are isomorphic to precisely those that were obtained in the previous sections. In this sense, the position representation built on solutions to the constraint equation that are separable in x_1 and x_2 is equivalent to the generalized angular momentum eigenstate representation (3.23) adopted in Refs. [3,22].

After the Cauchy completion, many of the Hilbert spaces will contain vectors that are not representable by functions of x_I . However, on certain subspaces of $\tilde{V}_{\text{pos},0}$ on which the representations of $\mathcal{A}_{\text{phy}}^{(*)}$ are isomorphic to those on V_{HH}^{\pm} , the inner product can be written as an integral formula which guarantees via standard L^2 theory that the vectors in the completion are representable by functions of x_I . We shall now demonstrate this. For concreteness, we only deal explicitly with the case in which the eigenenergies of the first oscillator are the positive half-integers. This means that we assume $\delta < \frac{1}{2}$, and the Hilbert spaces are isomorphic to $\mathcal{H}_{\text{HA}}^1$. The case in which the eigenenergies of the second oscillator are the positive half-integers and the Hilbert spaces are isomorphic to $\mathcal{H}_{\text{HA}}^2$ is analogous.

Let A and B be given complex numbers, not both equal to zero. Assuming $\delta < \frac{1}{2}$, consider the space $V_{\text{pos}}^{A,B} \subset \tilde{V}_{\text{pos},0}$ given by

$$V_{\text{pos}}^{A,B} = \text{Span} \left\{ \xi_k \in \tilde{V}_{\text{pos},0} \mid k = 0, 1, 2, \dots \right\} \subset V_{UU} \oplus V_{UV} \quad , \quad (5.6)$$

where

$$\begin{aligned} \xi_k(x_1, x_2) &= U\left(-k - \frac{1}{2}, \sqrt{2}x_1\right) \\ &\times \left[\frac{A}{\Gamma(k+1-2\delta)} U\left(2\delta - k - \frac{1}{2}, \sqrt{2}x_2\right) + B V\left(2\delta - k - \frac{1}{2}, \sqrt{2}x_2\right) \right] \quad . \end{aligned} \quad (5.7)$$

Here U and V are the parabolic cylinder functions as above. The representation of $\mathcal{A}_{\text{phy}}^{(*)}$ on $V_{\text{pos}}^{A,B}$ is isomorphic to that on V_{HA}^1 (4.7), with the isomorphism $\xi_k \mapsto \chi_{-\delta+k}$. From Eq. (4.8), the corresponding inner product on $V_{\text{pos}}^{A,B}$ is

$$(\xi_k, \xi_{k'}) = \frac{k!}{\Gamma(k+1-2\delta)} \delta_{k,k'} \quad . \quad (5.8)$$

We wish to demonstrate that, for certain values of A and B , this inner product can be written as an integral involving x_I .

Consider first the special case where 2δ is an integer; by assumption it is then negative or zero. We take $B = 0$ and $A = (1/\sqrt{\pi})$. Now, ξ_k reduce to products of (unconventionally normalized) harmonic oscillator eigenfunctions, and the inner product can be written as

$$(\xi_k, \xi_{k'}) = \int dx_1 dx_2 \overline{\xi_k(x_1, x_2)} \xi_{k'}(x_1, x_2) \quad . \quad (5.9)$$

Upon the Cauchy completion one therefore recovers a subspace of the conventional $L^2(\mathbf{R}^2)$ Hilbert space of two unconstrained harmonic oscillators. The role of the constraint has been just to choose the Hilbert subspace where the energy difference is equal to the integer 2δ .

Consider then the case of general $\delta < \frac{1}{2}$. As the quantum constraint in the position representation is of the Klein-Gordon form, a natural candidate for the inner product is

$$(\xi_k, \xi_{k'}) = i \int dx_1 \overline{\xi_k(x_1, x_2)} \overleftrightarrow{\partial}_2 \xi_{k'}(x_1, x_2) \quad , \quad (5.10)$$

where $\overleftrightarrow{\partial}_2 = \overrightarrow{\partial}_{x_2} - \overleftarrow{\partial}_{x_2}$ as usual. From the Wronskian relation of the parabolic cylinder functions [37] it follows that (5.10) holds, provided we set

$$A\bar{B} - B\bar{A} = \frac{i}{2} \quad . \quad (5.11)$$

Upon the Cauchy completion one thus arrives at a one-particle Klein-Gordon theory in two-dimensional Minkowski space with an external potential. The constants A and B satisfying the condition (5.11) specify the split between the positive and negative frequencies, and the Hilbert space is the one-particle Hilbert space associated with the positive frequency solutions. The integral kernel of the identity operator in this Hilbert space is the positive frequency Wightman function,

$$G^+(x_1, x_2; x'_1, x'_2) = \sum_k \frac{\xi_k(x_1, x_2) \overline{\xi_k(x'_1, x'_2)}}{(\xi_k, \xi_k)} \quad , \quad (5.12)$$

which satisfies

$$\xi(x_1, x_2) = i \int dx'_1 G^+(x_1, x_2; x'_1, x'_2) \overleftrightarrow{\partial}_2 \xi(x'_1, x'_2) \quad . \quad (5.13)$$

By construction, all choices for the constants A and B satisfying Eq. (5.11) lead to isomorphic quantum theories, in spite of the two real parameters (in addition to the overall phase) contained in these constants. However, if one interprets this theory as the positive frequency single-particle sector of a many-particle Klein-Gordon theory, one would usually take the viewpoint that not all the information of interest in the many-particle theory is contained in the operators in $\mathcal{A}_{\text{phy}}^{(*)}$. One could, for example, study the responses of particle detectors that couple to the Klein-Gordon field [38]. In this interpretation, different choices for A and B lead to different many-particle theories. A particularly simple choice is $A = \frac{1}{2}$, $B = -i/2$, in which case the ground state ξ_0 is the adiabatic vacuum [38]. Note that the adiabatic vacuum is among the one-parameter family of vacua that are symmetric about $x_2 = 0$, in the sense that

$$\xi_0(x_1, x_2) = \overline{\xi_0(x_1, -x_2)} \quad . \quad (5.14)$$

VI. COSMOLOGY AND THE NO-BOUNDARY PROPOSAL

As mentioned in the introduction, there exist spatially homogeneous cosmologies whose dynamics is, modulo certain global caveats, given by our action. Two such cosmologies are the positive curvature Friedmann model with a conformally or minimally coupled massless scalar field [18]; a third one is the vacuum Kantowski-Sachs model [19]. The quantum theories constructed in the previous sections can therefore be interpreted as quantizations of these cosmological models.

This cosmological interpretation raises new questions about the quantum theories. One such question is whether proposals that have been put forward for a “quantum state of the Universe” [28–30, 39–42] give quantum states that belong to any of our Hilbert spaces. In this section we wish to investigate this question for the no-boundary proposal of Hartle and Hawking [28–30]. For concreteness, we shall first focus on the positive curvature Friedmann model with a conformally coupled scalar field. The other two models will be discussed at the end of the section.

Recall that the no-boundary proposal of Hartle and Hawking [28–30] is a topological statement about the manifolds that are taken to contribute to the path integral in terms of which the wave function is defined. In the metric variables of general relativity one writes [12, 43, 44]

$$\Psi_{\text{NB}}(h_{ij}, \phi; \Sigma) = \sum_M \int \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi \exp[-I(g_{\mu\nu}, \Phi; M)] \quad , \quad (6.1)$$

where $I(g_{\mu\nu}, \Phi; M)$ is the (Euclidean) action of the gravitational field $g_{\mu\nu}$ and the matter fields Φ on the four-manifold M . The four-manifolds M are required to be compact with a boundary, such that their boundary is the three-surface Σ which appears in the argument of the wave function. The path integral is over metrics $g_{\mu\nu}$ and matter fields Φ on M which induce the values h_{ij} and ϕ on Σ . To give a meaning to this formal expression, additional input is needed. This would include specifying the four-manifolds M that contribute to the sum and their relative weights, and giving for each M a definition of the path integral. For further discussion of these general issues, see for example Ref. [43].

We now consider the positive curvature Friedmann model, whose (Lorentzian) metric is given by

$$ds^2 = \frac{2G}{3\pi} \left[-\tilde{N}^2(t) dt^2 + a^2(t) d\Omega_3^2 \right] \quad , \quad (6.2)$$

where $d\Omega_3^2$ is the metric on the unit three-sphere and G is Newton’s constant. The matter is taken to be a massless, conformally coupled scalar field Φ , homogeneous on the constant t surfaces. After the field redefinition

$$\begin{aligned} a &= x_2 \\ \Phi &= \left(\frac{3}{4\pi G} \right)^{\frac{1}{2}} \frac{x_1}{x_2} \\ \tilde{N} &= x_2 N \quad , \end{aligned} \quad (6.3)$$

the action of this model is given by Eq. (2.1), with $\delta = 0$ [29].

A delicate question in this field redefinition is the allowed range of the variables. The usual Hamiltonian formulation of general relativity in the metric variables assumes that the metric on the three-surfaces is positive definite. In the Friedmann model this means that the scale factor should be nonvanishing, and by convention one can take $a > 0$. Under this restriction the field redefinition (6.3) is nonsingular, and the equivalent restriction is $x_2 > 0$. In the previous sections, however, it was assumed that the coordinates x_I take all real values. We shall now proceed under the pretense that these global issues can be ignored and that the analysis of the previous sections is directly applicable to the cosmological situation. The validity of this pretense will be addressed at the end of the section.

Let us first consider the no-boundary proposal in a position representation. Now $\Sigma = S^3$, and we take the four-manifold in Eq. (6.1) to be the closed four-dimensional ball \bar{B}^4 [29]. The classical solution with the no-boundary data is well known. It is a real Euclidean solution, and in the Lorentzian conventions used above it can be written as

$$\begin{aligned} x_I(t) &= x_I^b e^t \\ N &= -i \end{aligned} \quad (6.4)$$

where x_I^b are constants. From Eqs. (6.2) and (6.3) it is seen that this is just a flat Euclidean metric with a constant Φ , expressed in a hyperspherical coordinate system where e^t is the radial coordinate. $t = -\infty$ corresponds to the coordinate singularity at the center of the \bar{B}^4 , and the boundary S^3 is at $t = 0$ with the induced values x_I^b for x_I . The action of this solution in terms of the boundary data is, in our Lorentzian conventions,

$$S_{class}(x_I^b) = \frac{i}{2} \left((x_1^b)^2 - (x_2^b)^2 \right) \quad (6.5)$$

Assuming that the no-boundary wave function is well approximated by the exponential of the negative of the classical Euclidean action, one obtains the estimate [29]

$$\Psi_{NB}(x_1, x_2) \sim P \exp \left(-\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \right) \quad (6.6)$$

where we have dropped the superscript b . The prefactor P is assumed to be slowly varying compared with the exponential factor.

Two comments here are in order. Firstly, it appears not to be known whether in this model there exists a set of no-boundary “initial” conditions for a minisuperspace path integral such that these conditions could be stated in terms of acceptable combinations of the initial values of x_I and p_I . A general discussion of this kind of conditions can be found in Refs. [44,45] (see also Ref. [46]). We shall therefore not attempt to define a no-boundary wave function in the position representation beyond the above semiclassical estimate.

Secondly, we have here followed Ref. [29] in choosing the conventional sign in the “Wick rotation” which relates the Lorentzian and Euclidean theories. This amounts to the relation $iS = -I$ between the Lorentzian and Euclidean actions. For a discussion of this point, see Refs. [43,44].

Return now to those $\delta = 0$ quantum theories of section V where we saw that all states in the Hilbert space are representable by functions of x_I . Do any of these Hilbert spaces admit wave functions of the form (6.6)? The answer is clearly affirmative, as is seen from the asymptotic properties of the parabolic cylinder functions [37]: all the Hilbert spaces

built on $V_{\text{pos}}^{A,B}$ (5.6) satisfying the condition (5.11) contain such wave functions. Further, the wave function that arguably most closely conforms to Eq. (6.6) is the ground state ξ_0 annihilated by \hat{J}_- . We have thus found Hilbert spaces in which the ground state exhibits the asymptotic behavior associated with the semiclassical no-boundary wave function.

Note that although the leading order exponential term at $x_2 \rightarrow \infty$ in these ground state wave functions is real, the wave functions are nevertheless genuinely complex-valued. Recovering such complex-valued wave functions from the no-boundary integral (6.1) by suitable complex integration contours has been discussed for example in Refs. [43,44,47,48].

None of the Hilbert spaces built on the counterparts of $V_{\text{pos}}^{A,B}$ with x_1 and x_2 interchanged admit wave functions with the semiclassical form (6.6). The same is true for the Hilbert space built on $V_{\text{pos}}^{(1/\sqrt{\pi}),0}$. Note, however, that states in the Hilbert space built on $V_{\text{pos}}^{(1/\sqrt{\pi}),0}$ have been suggested as representing wormhole wave functions [49].

We now turn to the holomorphic representations. Investigating the no-boundary proposal in these representations is very similar to the analogous problem in systems involving fermions [50–53].

We anticipate that the no-boundary wave function can be written as the integral

$$\Psi_{\text{NB}} = \int \mathcal{D}(z_I, \bar{z}_I, N) \exp(iS) \quad , \quad (6.7)$$

where the arguments of Ψ_{NB} are the appropriate ones for the chosen representation, and S consists of the integral term in the actions (3.25) and (4.16) with some appropriate boundary terms. As in the metric variables, we wish to define the integral to be over fields living on the four-manifold \bar{B}^4 . For simplicity, we continue to employ a Lorentzian notation. Whether the configurations to be integrated over are Lorentzian or Euclidean or complex would depend on the precise definition to be given to the integral.

We would like to define the expression (6.7) to be compatible with the classical system in the sense that the classical solutions to the corresponding boundary value problem be well-defined. In a Berezin-type path integral for fermionic systems, such a condition has been applied to the no-boundary proposal in Refs. [50–53]. We shall adopt the analogous condition here. To make this precise, recall that Eqs. (2.3) can be inverted to give

$$\begin{aligned} x_I &= \frac{1}{\sqrt{2}}(z_I + \bar{z}_I) \\ p_I &= \frac{i}{\sqrt{2}}(z_I - \bar{z}_I) \quad . \end{aligned} \quad (6.8)$$

We therefore define a solution to the holomorphic equations of motion to be regular if the corresponding four-dimensional gravity and matter fields computed from Eqs. (6.3) and (6.8) are regular on \bar{B}^4 .

Consider now the general solution to the holomorphic equations of motion,

$$\begin{aligned} z_1 &= C_1 e^{i\tau} \\ \bar{z}_1 &= C_2 e^{-i\tau} \\ z_2 &= C_3 e^{-i\tau} \\ \bar{z}_2 &= C_4 e^{i\tau} \quad , \end{aligned} \quad (6.9)$$

where C_i are constants, $\tau = \int^t N(t') dt'$, and N is allowed to take general complex values. With the choice $N = -i$, a regular classical solution is obtained by setting $C_2 = C_3 = 0$. This is the holomorphic counterpart of the solution (6.4); note that both the metric and the matter field in this solution are complex-valued for generic values of C_1 and C_4 . $t = 0$ gives again the boundary of \bar{B}^4 , and $t = -\infty$ gives the coordinate singularity at the center. The corresponding classical boundary value problem in terms of z_I and \bar{z}_I consists of setting the initial values of \bar{z}_1 and z_2 to zero, and specifying freely the final values of z_1 and \bar{z}_2 . The action appropriate for this boundary value problem is given by Eq. (4.16). One is therefore led to expect that the no-boundary integral is given by the amplitude $G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2)$ (4.15), with $\bar{\beta}_1 = 0 = \beta_2$. In particular, this means that the no-boundary integral gives a wave function in the holomorphic-antiholomorphic representation. This is analogous to what happens in the massless fermionic systems in Refs. [50–52].

Let us try to evaluate this no-boundary integral. Consider first a semiclassical estimate. The classical action (4.16) evaluated on a solution for which the initial values of \bar{z}_1 and z_2 vanish is equal to zero, and the semiclassical estimate to the integral therefore consists entirely of a prefactor. Such an estimate is compatible with many states in the Hilbert space $\mathcal{H}_{\text{HA}}^1$, arguably most closely with the ground state $\chi_0(z_1, \bar{z}_2) = 1/\bar{z}_2$ annihilated by \hat{J}_- . For concreteness, we concentrate here on $\mathcal{H}_{\text{HA}}^1$; an analogous discussion holds for $\mathcal{H}_{\text{HA}}^2$.

Consider then the exact amplitude evaluated in subsection IV B. The above discussion implies that the no-boundary state should be given by

$$\Psi_{\text{NB}}(z_1, \bar{z}_2) = \lim_{\substack{\bar{\beta}_1 \rightarrow 0 \\ \beta_2 \rightarrow 0}} \Psi_{\bar{\beta}_1, \beta_2}(z_1, \bar{z}_2) \quad , \quad (6.10)$$

where $\Psi_{\bar{\beta}_1, \beta_2}$ is given by (4.22). Unfortunately, even though $\Psi_{\bar{\beta}_1, \beta_2}$ is in $\mathcal{H}_{\text{HA}}^1$ for $|\bar{\beta}_1/\beta_2| < 1$, the limit (6.10) does not yield a vector in $\mathcal{H}_{\text{HA}}^1$, as it is seen from Eqs. (4.19) and (4.23) that $(\Psi_{\bar{\beta}_1, \beta_2}, \Psi_{\bar{\beta}_1, \beta_2})$ diverges no matter how $\bar{\beta}_1$ and β_2 are taken to zero.

Thus, although a semiclassical estimate for the no-boundary integral is compatible with some states in the Hilbert space $\mathcal{H}_{\text{HA}}^1$, and arguably most closely with the ground state, the exact path measure of subsection IV B does not yield a well-defined no-boundary wave function. An attempt to improve this situation might be to argue that before taking the limit in Eq. (6.10), it is legitimate to renormalize $\Psi_{\bar{\beta}_1, \beta_2}$ by a factor which only depends on $\bar{\beta}_1$ and β_2 . A suitable choice for this renormalization factor and a suitable prescription for taking $\bar{\beta}_1$ and β_2 to zero then produces the ground state wave function, as is easily seen from Eq. (4.24). It is, however, not clear whether a prescription of this kind can be justified from the path integral.

The above discussion was based on the regular classical solution in which $N = -i$ and $C_2 = C_3 = 0$ in (6.9). There exists a second regular solution, obtained with $N = -i$ and $C_1 = C_4 = 0$. In this solution, t ranges from $t = 0$ at the boundary of \bar{B}^4 to $t = +\infty$ at the coordinate singularity at the center of \bar{B}^4 . The no-boundary path integral with the boundary data compatible with this solution gives a wave function in the antiholomorphic-holomorphic representation. It was seen in section IV that the Hilbert spaces obtained in the antiholomorphic-holomorphic representation with $\delta = 0$ are isomorphic to those obtained in the holomorphic-antiholomorphic representation. This implies that a discussion

of the antiholomorphic-holomorphic no-boundary wave function proceeds exactly as in the holomorphic-antiholomorphic case above.

Note that once the relation between the Lorentzian and Euclidean theories has been fixed to be the conventional one, the choice between a classical solution defined for $-\infty < t \leq 0$ or $0 \leq t < \infty$ does not make a difference in the semiclassical estimate for the no-boundary wave function in the metric variables. In the holomorphic variables, however, this choice picks respectively the holomorphic-antiholomorphic or antiholomorphic-holomorphic representation. A similar observation was made in a supersymmetric system in Ref. [52].

Return now to the restriction $x_2 > 0$, which has been ignored so far. In the Dirac quantization, most of the analysis of the previous sections appears to go through unperturbed when this restriction is introduced, both in the holomorphic representations and the position representation. Notice in particular that the allowed values of the J 's (2.5) are unaffected. The only step where the restriction seems to make an essential difference is when one attempts to define integral formulas for the inner product. The formula (5.10) for the Klein-Gordon inner product in the position representation remains nevertheless valid. The restriction $x_2 > 0$ appears also to be consistent with all the steps in finding the semiclassical estimates to the no-boundary wave functions, both in the position representation and the holomorphic representations.

On the other hand, the step that seems unlikely to remain justified under the restriction $x_2 > 0$ is the construction of the exact holomorphic path measures in subsection IV A. Our method relied on introducing first an unconstrained path integral for each oscillator, and finally integrating over the Lagrange multiplier to enforce the constraint. The restriction $x_2 > 0$ would affect the unconstrained path integral for the second oscillator, and hence most likely the final result.

To end this section, let us briefly discuss two other cosmological models whose dynamics is described by our action. First, consider again the positive curvature Friedmann model (6.2), but now with a massless, minimally coupled scalar field ϕ , homogeneous on the constant t surfaces. The field redefinition mapping this system to our model with $\delta = 0$ is [18]

$$\begin{aligned} a &= \sqrt{x_2^2 - x_1^2} \\ \phi &= \left(\frac{3}{4\pi G}\right)^{\frac{1}{2}} \operatorname{artanh}\left(\frac{x_1}{x_2}\right) \\ \tilde{N} &= \sqrt{x_2^2 - x_1^2} N \ , \end{aligned} \tag{6.11}$$

and the restriction for the coordinates is $x_2 > |x_1|$. The Dirac quantization proceeds as above; however, the formula (5.10) for the Klein-Gordon inner product is now not compatible with the restriction. Similarly, the semiclassical estimates to the no-boundary wave function with the four-manifold \bar{B}^4 are exactly as above, implying compatibility of the Hilbert spaces and the semiclassical no-boundary wave function. An attempt to find an exact holomorphic no-boundary wave function using the measure of subsection IV B faces the same difficulty as with the conformally coupled scalar field.

Finally consider the vacuum Kantowski-Sachs model. The metric is given by

$$ds^2 = \frac{G}{2\pi} \left[-\tilde{N}^2(t) dt^2 + a^2(t) d\chi^2 + b^2(t) \Omega_2^2 \right] \ , \tag{6.12}$$

where $d\Omega_2^2$ is the metric on the unit two-sphere and χ is an angular coordinate with period 2π . The field redefinition leading to our action with $\delta = 0$ is [19]

$$\begin{aligned} a &= \frac{x_2 - x_1}{x_2 + x_1} \\ b &= \frac{1}{2}(x_2 + x_1)^2 \\ \tilde{N} &= (x_2 + x_1)^2 N \quad , \end{aligned} \tag{6.13}$$

and the restriction is again $x_2 > |x_1|$. The Dirac quantization proceeds thus as in the Friedmann model with the minimally coupled scalar field. Now, however, there are two relevant four-manifolds that admit no-boundary type classical solutions [54,55]. For the four-manifold $\bar{B}^3 \times S^1$, the analysis proceeds as for the Friedmann model with a minimally coupled field. For the four-manifold $\bar{B}^2 \times S^2$, on the other hand, the semiclassical estimate to the no-boundary wave function in the metric variables is more complicated [44,54–56], and it is not obvious how to formulate the corresponding variational problem in the holomorphic variables. We shall not attempt to discuss this manifold further.

VII. CONCLUSIONS AND DISCUSSION

In this paper we have discussed the Dirac quantization and path integral quantization of a simple quantum mechanical system with a quadratic Hamiltonian constraint. The system consists of two harmonic oscillators with identical frequencies, and the dynamics is given by a Hamiltonian constraint which sets the energy difference of the oscillators equal to a prescribed real number. In the Dirac quantization we employed a class of representations which generalize Bargmann’s coherent state representation for an ordinary unconstrained oscillator. In each of our representations an inner product was uniquely determined, under certain assumptions, by imposing Hermitian conjugacy relations on a suitable operator algebra. The sets of the Hilbert spaces obtained in the different representations were found to have drastically different properties; in particular, some of these Hilbert spaces contradicted the classical expectations for the spectra of certain operators. The analogous Dirac quantization in a conventional position representation was shown to give Hilbert spaces that are isomorphic to those obtained in the coherent state representations in which the spectra of the operators do not violate classical expectations. Carefully defined coherent state path integrals were shown to yield the matrix elements of the identity operator in the Hilbert spaces obtained in Dirac’s quantization, with respect to an overcomplete basis of representation-dependent generalized coherent states. Finally, the results were interpreted as quantizations of a class of cosmological models. Both in the position representation and in the relevant coherent state representations, the “ground state” in the appropriate Hilbert spaces was shown to have the semiclassical behavior expected of the no-boundary wave function of Hartle and Hawking.

Our Dirac quantization was carried out following closely the algebraic quantization program of Ashtekar [3]. This program was previously applied to our model in Refs. [3,22], and we took our classical Poisson bracket algebras and quantum commutator algebras to be directly the same as in that earlier work. The point where we differed from the earlier work

was the choice of the vector spaces on which these algebras, most importantly the physical operator algebra $\mathcal{A}_{\text{phy}}^{(*)}$, are represented. Whereas $\mathcal{A}_{\text{phy}}^{(*)}$ in Refs. [3,22] was represented on a vector space spanned by generalized angular momentum eigenstates, we took the vector space to consist of certain functions of two real or complex variables. After finding the inner product, the Cauchy completion was seen to lead precisely to the set of abstract Hilbert spaces obtained in Refs. [3,22]; however, some of these spaces contained vectors that are no longer representable by functions. We then restricted the attention to the Hilbert spaces where all the vectors are representable by functions, and found that this set contains the most physically interesting spaces. In particular, this set contains the spaces where the spectrum of the operator \hat{J}_0 (3.6c) is bounded below, in agreement with the lower bound for the corresponding classical function J_0 (2.5c).

Our representations on functions of complex variables generalized Bargmann's coherent state representation for a single unconstrained oscillator in that we allowed the functions to be either analytic or antianalytic in each of the arguments. Analyticity in both arguments led precisely to those spaces where the spectrum of \hat{J}_0 satisfies the classical expectations, whereas antianalyticity in both arguments led only to spaces where these classical expectations are violated. Among the mixed representations, on the other hand, the spaces where the classical expectations for \hat{J}_0 are satisfied are distributed so that the eigenenergies of the analytically represented oscillator are always the positive half-integers.

Although the representations thus exhibit a clear disparity between analyticity and antianalyticity, it is interesting that this disparity is not as severe as in the case of an unconstrained harmonic oscillator. For the unconstrained oscillator, the standard Bargmann theory is recovered by considering the operator algebra generated by the operators $\{\hat{z}, \hat{\bar{z}}, \hat{\mathbb{I}}\}$ with commutation relations analogous to Eqs. (3.1), with a representation analogous to that in Eqs. (3.3) on the vector space $\text{Span}\{z^k \mid k = 0, 1, 2, \dots\}$. The standard inner product is uniquely recovered by imposing the Hermitian conjugacy relation $\hat{z}^\dagger = \hat{\bar{z}}$. If one instead tries to represent this algebra antianalytically, in analogy with Eqs. (3.21) on a vector space spanned by powers of \bar{z} , there will be no inner product compatible with the Hermitian conjugacy relation. An inner product can however be found if one adopts the antianalytic representation on a vector space involving not functions but (anti)holomorphic distributions [57].

In the case where the energy difference 2δ is an integer, it is clear how to build a quantum theory for our system starting from the conventional quantum theory of two unconstrained oscillators. Our method recovered quantum theories for arbitrary real values for 2δ , and for integer values of 2δ one of our theories was indeed isomorphic to the expected theory. It is natural to ask whether the theories with integer values of 2δ are distinguished by some special properties. One way to look at this is to recall that our Hilbert spaces carry representations of the Lie algebra of $SO(2, 1)$. By exponentiation this means that the spaces carry representations of the universal covering group of $SO_c(2, 1)$, but not necessarily those of $SO_c(2, 1)$ or its double cover $SU(1, 1)$. (Here $SO_c(2, 1)$ stands for the connected component of $SO(2, 1)$.) However, our spaces with integer eigenvalues of \hat{J}_0 do carry unitary representations of $SO_c(2, 1)$, and those with half-integer eigenvalues of \hat{J}_0 carry unitary representations of $SU(1, 1)$ [31]. Within the spaces in which the spectrum of \hat{J}_0 is bounded below, this means that representations of $SO_c(2, 1)$ are obtained in $\mathcal{H}_{\text{HH}}^+$ with 2δ odd, and

representations of $SU(1, 1)$ are obtained in $\mathcal{H}_{\text{HH}}^+$ with 2δ even.

Another way to look at this question is to consider a modified theory where we add to the system a true classical Hamiltonian equal to J_0 . The constraint remains first class [2], and from Eqs. (2.6) it is seen that the classical motion is periodic in t with period 2π . Taking the quantum Hamiltonian to be \hat{J}_0 (3.6c), the Hilbert spaces are the same as before, but now there is a Schrödinger equation giving these spaces nontrivial dynamics. For concreteness, let us only consider the spaces $\mathcal{H}_{\text{HH}}^\pm$, in which the spectrum of \hat{J}_0 is bounded below. The Schrödinger evolution in $\mathcal{H}_{\text{HH}}^\pm$ is not periodic for general values of δ ; however, it is periodic with period 2π up to the phase $e^{i\pi(-1\mp 2|\delta|)}$. Integer values of 2δ (which by definition of $\mathcal{H}_{\text{HH}}^\pm$ only can occur with the upper sign) are distinguished by making this phase real. A related discussion involving a parity operator has been given in Refs. [3, 22].

It is also possible to do the path integrals of subsections III B and IV B in the presence of J_0 as the true Hamiltonian. The propagation amplitudes analogous to Eqs. (3.29) and (4.19) are

$$G_\pm(\alpha_I; \bar{\beta}_I; t) = e^{-it/2} \left(\frac{\alpha_1 \bar{\beta}_1}{\alpha_2 \bar{\beta}_2} \right)^\delta I_{\pm 2|\delta|} \left(2e^{-it/2} \sqrt{\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2} \right) \quad (7.1)$$

and

$$G(\alpha_1, \bar{\alpha}_2; \bar{\beta}_1, \beta_2; t) = \frac{\Gamma(1 - 2\delta)}{\left(\bar{\alpha}_2 \beta_2 e^{it/2} - \alpha_1 \bar{\beta}_1 e^{-it/2} \right)^{1-2\delta}}. \quad (7.2)$$

These amplitudes are readily recognized as the matrix elements of the time evolution operator $\exp(-it\hat{J}_0)$. Note that they are periodic in t with period 2π up to the δ -dependent phase.

As has been emphasized in Ref. [22], it is remarkable that there emerged no “quantization condition” for the allowable values of δ . One is inclined to relate this to the fact that the reduced phase space of the classical system has infinite volume, as was seen at the end of section II. When a similar analysis is carried out in a model where the energy difference constraint is replaced by an energy sum constraint, so that in particular the reduced phase space has finite volume, one indeed finds that there is a quantization condition for the allowed values of the energy sum [22, 58–60]. The relation of the different representations in the energy sum model is very similar to that in the energy difference model, and also coherent state path integrals can be constructed very much as in the energy difference model. We shall briefly outline these results in the appendix.

On the other hand, both the energy difference model and the energy sum model are rather special among all quadratic super-Hamiltonians in that the frequencies of the two oscillators were fine-tuned to be equal. It would be of interest to understand whether the approach in this paper could be adapted to the more general case where the frequencies are different. One might anticipate problems at least when the ratio of the two frequencies is irrational, since the classical motion in that case is highly chaotic [21].

In sections III and IV, we defined the formal holomorphic path integral expressions so as to obtain quantities that are related to the Hilbert spaces. The definition involved several delicate points, perhaps most crucially the assumption that the integral over the

Lagrange multiplier can be pulled outside the $\mathcal{D}(z_I, \bar{z}_I)$ integrals, and that the contour for the Lagrange multiplier can then be chosen complex. A similar method has been used in quantum cosmology, and the possible caveats of this method discussed in that context in Refs. [44,46] may well be relevant also here.

It might be of interest to investigate in more detail also conventional configuration space path integrals in our system, and their relation to the Hilbert spaces obtained in the position representation in section V. Some issues involving the definition of such integrals in this model have been discussed in Ref. [61].

In section VI we saw that our model can be reinterpreted as a quantization of certain cosmological models, provided the range of the configuration space variables is suitably restricted. Whilst such a restriction seemed to have little effect on the Dirac quantization, one nevertheless expects that the operators of physical interest in the quantized cosmologies should be constructed in a way which takes into account the ranges of the variables. One suggestion to construct such operators is via the “evolving constants of motion” [22,59,62–64] (see, however, the discussion in Ref. [65]).

The cosmological restriction on the range of the configuration space variables was not compatible with the construction of the exact holomorphic path measures in subsection IV A. In view of this, one is tempted to regard our affirmative results about the compatibility of the Hilbert spaces and the semiclassical no-boundary wave function as more interesting than the failure of the measure of subsection IV A to yield a well-defined exact no-boundary wave function. In the Friedmann model with a conformally coupled scalar field, it might be possible to investigate this issue further in a representation where the wave functions are analytic on a half-plane [66].

In the cosmological interpretation, our approach bears some similarity to the quantization of spatially homogeneous cosmologies in the connection representation of Ashtekar’s variables [3,67,68]. There are, however, two significant differences.

Firstly, recall that the functions z_I and \bar{z}_I on the classical phase space take arbitrary complex values, or under the cosmological restrictions arbitrary values in certain open sets in the complex plane. This makes it natural to look for a quantum representation where the wave functions are analytic or antianalytic functions of complex variables. On the other hand, when spatially homogeneous cosmologies are written in terms of Ashtekar’s variables, the real part of the Ashtekar connection A is a constant function over the classical Lorentzian phase space [67–69]. The value of this constant is determined by the homogeneity type; for example in Bianchi type I the real part of A simply vanishes. One would therefore expect the A -representation of the quantum theory to be more analogous to a conventional momentum representation than to a coherent state representation, and this makes it unclear whether the wave functions should be analytic in A . A specific proposal for analyticity was made in Ref. [68].

Secondly, recall that for our semiclassical estimate to the no-boundary wave function we had to solve a classical boundary value problem. In the mixed representations, it was found that the relevant boundary value problem had a solution for arbitrary complex values of the boundary data compatible with the restriction for the range of the variables. In terms of Ashtekar’s variables, however, the corresponding classical boundary value problem is more problematic. Consider for concreteness the Euclidean version of the theory, where A classically takes generic real values, and suppose that the cosmological constant vanishes.

In the cosmological models that have been investigated, the no-boundary classical solutions are such that four-dimensional regularity at the “center” of the four-manifold fixes the components of the “initial” connection to certain numerical values [45]. As Ashtekar’s super-Hamiltonian consists of a pure kinetic term with a Lorentz signature supermetric, this means that solutions to the corresponding boundary value problem with a specified boundary connection exist only when this boundary connection lies on the null cone (with respect to the supermetric) of the “initial” connection. Thus, for a given generic boundary connection, solutions to the boundary value problem do not exist. It is unclear what this should be taken to imply for the semiclassical estimate to a no-boundary wave function in the connection representation of the Ashtekar variables.

One should emphasize that these two points rely heavily on the restriction of the Ashtekar variables to spatially homogeneous cosmologies. It is not clear whether these points remain relevant when discussing the connection representation without such symmetry assumptions.

Finally, we would like to briefly discuss the relation of semiclassical methods to our Dirac quantization. In much of the work on quantum cosmology, physically interesting solutions to the quantum Hamiltonian constraint have been sought in the form of a Born-Oppenheimer approximation, the celebrated result being the derivation of the functional Schrödinger equation of quantum field theory on curved spacetime from the Wheeler-DeWitt equation [70]. This method has been criticized [65], among other things on the grounds that one does not assume the original wave function to lie in a Hilbert space, but rather one only later introduces additional approximate normalization conditions for parts of the wave function to recover physical predictions. It therefore remains unclear what the theory is that the approximation seeks to approximate. This leads to problems when discussing for example superpositions of solutions.

Suppose now that one wishes to use the Born-Oppenheimer method in our model. Does one recover physical predictions that agree with those in any of our exact Hilbert spaces?

Mimicking the usual approach in the cosmological case, let us consider quantum theory in the position representation. We wish to assume that the second oscillator dominates the dynamics in some appropriate sense, and that the first oscillator can be treated as a small perturbation. We therefore expect the energy of the second oscillator to be large compared with that of the first one, so that δ is large and negative. To implement this, we write $\delta = -\frac{1}{4}M\gamma$, where γ is considered a fixed positive number, and the limit of interest will be that of large positive M . We also perform the rescaling $x_2 \rightarrow M^{1/2}x_2$, $p_2 \rightarrow M^{-1/2}p_2$; this is just a canonical transformation and in no way changes the physics. The classical constraint is then given by

$$\mathcal{H} = \frac{1}{2} \left[-\frac{p_2^2}{M} + M(\gamma - x_2^2) \right] + \frac{1}{2}(p_1^2 + x_1^2) \quad . \quad (7.3)$$

The quantum constraint obtained by the substitution $p_I^2 \rightarrow -\partial^2/(\partial x_I^2)$ is of a form in which the Born-Oppenheimer approximation can be implemented by the standard formal expansion at $M \rightarrow \infty$ [65,70]. In the next-to-leading order in this expansion one recovers the ordinary Schrödinger equation for the first oscillator, and if one requires the solutions to this equation to be normalizable in the usual Schrödinger inner product, one obtains the prediction that the energy eigenvalues of the first oscillator are just the positive half-integers [71]. This prediction is in exact agreement with that obtained in our Hilbert space $\mathcal{H}_{\text{HA}}^1$.

On the other hand, it has been suggested that in the quantum cosmological context one could derive quantum gravitational corrections to quantum field theory in curved spacetime by carrying out the Born-Oppenheimer expansion beyond the next-to-leading order [71,72]. When applied to our model, this method has been argued [71] to be consistent with a higher order correction in $1/M$ to the half-integer eigenenergies of the first oscillator. Such a correction would clearly no longer be in agreement with the exact Hilbert space $\mathcal{H}_{\text{HA}}^1$. (Note, however, that the argument in the form given in Ref. [71] relies on introducing in Eq. (20) therein a quantity $F(q)$ which is formally divergent.)

Thus, the Born-Oppenheimer approximation gives, at least in the next-to-leading order, predictions that are consistent with those obtained in one of our exact Hilbert spaces. One may see this as encouraging for the Born-Oppenheimer approximation, and one might wish to use the position representation realizations of this Hilbert space that were constructed in section V to further examine the range of validity of the approximation. An open question in such an approach would however be how to distinguish effects due to the the simplicity of our model from potentially more general effects.

ACKNOWLEDGMENTS

I would like to thank Abhay Ashtekar, Claus Kiefer, Don Marolf, Jonathan Simon, Tejinder Singh and Chrysis Soteriou for discussions on various parts of this work. I would especially like to thank Ranjeet Tate for numerous discussions and for making a preliminary version of Ref. [22] available. This work was supported in part by the NSF grant PHY90-16733 and by research funds provided by Syracuse University.

APPENDIX: ENERGY SUM MODEL

The methods used in this paper can be applied without essential changes in a model in which the energy difference constraint (2.2) is replaced by the energy sum constraint [22,58–60]

$$\tilde{\mathcal{H}} = \frac{1}{2} (p_1^2 + p_2^2 + x_1^2 + x_2^2) - E \quad , \quad (\text{A1})$$

where E is an arbitrary real number. In this appendix we shall outline the results.

The physical operator algebra $\tilde{\mathcal{A}}_{\text{phy}}^{(*)}$ is now generated by the set $\{\hat{L}_{\pm}, \hat{L}_0, \hat{\mathbb{I}}\}$, where

$$\begin{aligned} \hat{L}_+ &= \hat{z}_1 \hat{\bar{z}}_2 \\ \hat{L}_- &= \hat{z}_2 \hat{\bar{z}}_1 \\ \hat{L}_0 &= \frac{1}{2} (\hat{z}_1 \hat{\bar{z}}_1 - \hat{z}_2 \hat{\bar{z}}_2) \quad , \end{aligned} \quad (\text{A2})$$

and the commutators are

$$\begin{aligned} [\hat{L}_+, \hat{L}_-] &= 2\hat{L}_0 \\ [\hat{L}_0, \hat{L}_{\pm}] &= \pm \hat{L}_{\pm} \quad . \end{aligned} \quad (\text{A3})$$

The \hat{L} 's are recognized as the generators of the Lie algebra of $SU(2)$. With the factor ordering

$$\hat{\mathcal{H}} = \hat{z}_1 \hat{z}_1 + \hat{z}_2 \hat{z}_2 + (1 - E) \hat{\mathbb{I}} \quad , \quad (\text{A4})$$

the constraint $\hat{\mathcal{H}}\psi = 0$ is easily solved in the doubly holomorphic, doubly antiholomorphic, and mixed representations, and the irreducible representations of $\tilde{\mathcal{A}}_{\text{phy}}^{(*)}$ on vector spaces analogous to those in the energy sum model are easily found. The Hermitian conjugacy relations inherited from the complex conjugation properties of the classical counterparts of the \hat{L} 's are

$$\hat{L}_{\pm}^{\dagger} = \hat{L}_{\mp} \quad , \quad \hat{L}_0^{\dagger} = \hat{L}_0 \quad . \quad (\text{A5})$$

In the doubly holomorphic representation, one obtains quantum theories only for positive integer values of E , and the Hilbert spaces are just the usual finite dimensional spaces carrying unitary representations of the group $SU(2)$. This reproduces the result of Refs. [22,58–60]. Taking E to be a positive integer, the path integral analogous to that in subsection III A yields the amplitude

$$G(\alpha_I; \bar{\beta}_I) = \frac{1}{(E-1)!} (\alpha_1 \bar{\beta}_1 + \alpha_2 \bar{\beta}_2)^{E-1} \quad , \quad (\text{A6})$$

where the integration contour has been chosen as for integer values of 2δ in subsection III A. This amplitude gives the matrix elements of the identity operator in the relevant Hilbert spaces in the overcomplete basis formed by the displacement operator coherent states for $SU(2)$ [26,27].

In the doubly antiholomorphic representation one recovers quantum theories only for negative integer values of E . A path integral can be constructed as above. These Hilbert spaces carry unitary representations of $SU(2)$ that are isomorphic to those in the doubly holomorphic representation; however, the classical system has no solutions for negative E . The relation between the doubly holomorphic and doubly antiholomorphic representations is therefore very similar to that in the energy difference model.

In the mixed representations there exist no inner products that would satisfy the Hermitian conjugacy relations (A5), and no Hilbert spaces are obtained.

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